Lesson 29: Double Integrals

Recall from Lesson 2, the formula for average value:

For $f(x)$ defined on $[a,b]$, the average value of $f(x)$ on $[a,b]$ is

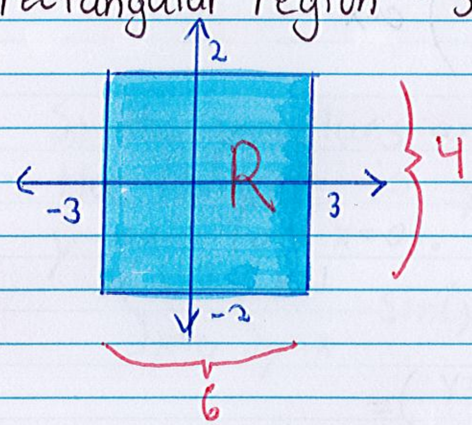
$$f_{AVE}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The multivariable average value formula follows:

For $f(x,y)$ defined on a region, R, the average value of $f(x,y)$ over the region, R, is given by

$$f_{AVE}(x,y) = \frac{1}{A}\iint_R f(x,y)\,dA$$ where A is the area of R.

Example 1: Find the average of $f(x,y) = 12 - x^2 - y^2$ in a rectangular region $-3 \le x \le 3$, $-2 \le y \le 2$.



First draw the region. With the drawing, find the area of R.

Area of R $= 6 \times 4 = 24$

Note we are given the bounds for our integral. So

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\int_{y=-2}^{y=2}(12-x^2-y^2)\,dy\,dx$$

Now integrate.

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\left(\left(12y - x^2y - \frac{y^3}{3}\right)\right]_{y=-2}^{y=2} dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(12(2) - 2x^2 - \frac{2^3}{3} - \left(12(-2) - x^2(-2) - \frac{(-2)^3}{3}\right)\right) dx$$

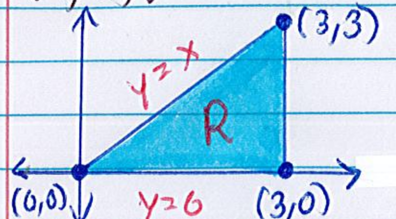
$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(24 - 2x^2 - \frac{8}{3} + 24 - 2x^2 - \frac{8}{3}\right) dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(\frac{128}{3} - 4x^2\right) dx$$

$$= \frac{1}{24}\left(\frac{128}{3}x - \frac{4x^3}{3}\right)\Big]_{x=-3}^{x=3}$$

$$= \frac{1}{24}\left(\frac{128}{3}(3) - \frac{4(3)^3}{3} - \left(\frac{128}{3}(-3) - \frac{4(-3)^3}{3}\right)\right) = \frac{28}{3}$$

Example 2: Find the average value of $f(x,y) = x^2 + 2xy + y^2$ in the triangular region with vertices $(0,0)$, $(3,0)$, and $(3,3)$.



First draw the region. With drawing, find the area of R.

$$\text{Area of } R = \frac{1}{2}(3)(3) = \frac{9}{2}$$

Next, we need to determine the bounds for our integral. The x-values are $0 \le x \le 3$. As for y, we see that y lies between the x-axis ($y=0$) and the line $y=x$. So

$$f_{AVE} = \frac{1}{9/2}\int_{x=0}^{x=3}\int_{y=0}^{y=x}(x^2 + 2xy + y^2)\,dy\,dx$$

$$= \frac{2}{9}\int_{x=0}^{x=3}\left(\left(x^2 y + \frac{2xy^2}{2} + \frac{y^3}{3}\right)\Bigg]_{y=0}^{y=x}\right)dx$$

$$= \frac{2}{9}\int_{x=0}^{x=3}\left(x^3 + x\cdot x^2 + \frac{x^3}{3}\right)dx$$

$$= \frac{2}{9}\int_{x=0}^{x=3}\frac{7}{3}x^3\,dx$$

$$= \frac{2}{9}\cdot\frac{7}{3}\,\frac{x^4}{4}\Bigg]_{x=0}^{x=3}$$

$$= \frac{2}{9}\cdot\frac{7}{3}\cdot\frac{3^4}{4} = \frac{21}{2}$$

Recall from Last Time, given a function $z = f(x,y)$ and a region, R, in the xy-plane, we have

$$\iint_R f(x,y)\,dA$$

To solve these integrals, we start by drawing the region. We do this to determine the bounds of our integrals. And then integrate.

But sometimes the integral we obtain can't integrate it. Mainly because we don't know it's antiderivative. So when should you use $dx\,dy$ or $dy\,dx$ is vital in these problems.

i.e. This is where switching the order of integration comes in. Given an integral with dxdy, we can switch it to dydx (and vice versa) via the drawing of the region.

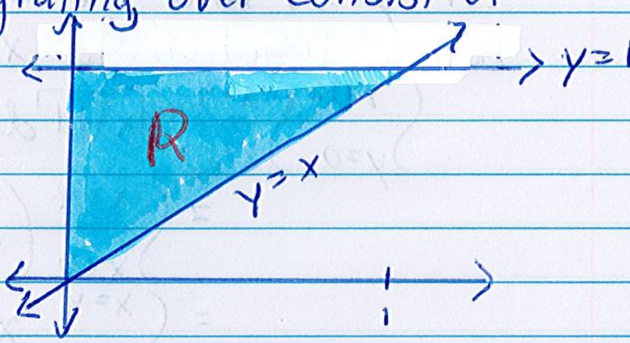
It's easier to see through some examples.

Example 3: Compute

ⓐ $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2)\,dy\,dx$

So the region we are integrating over consist of

$0 \le x \le 1, \; x \le y \le 1$



Note: $y=x$ is the bottom function, while $y=1$ is the top.

So the y-values are $0 \le y \le 1$. The x-values we see that the largest is $y=x$ (or $x=y$) and smallest is the y-axis (or $x=0$). So $0 \le x \le y$. Hence

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2)\,dy\,dx = \int_{y=0}^{y=1} \int_{x=0}^{x=y} \sin(y^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} \left(\sin(y^2) \cdot x \Big]_{x=0}^{x=y} \right) dy$$

$$= \int_{y=0}^{y=1} y \sin(y^2)\,dy$$

$$\xrightarrow[\substack{du = 2y\,dy \\ du/2 = y\,dy}]{u=y^2} \int \sin(u) \frac{du}{2}$$

$$= -\frac{1}{2} \cos(u)$$

$$= -\frac{1}{2} \cos(y^2) \Big]_{y=0}^{y=1}$$

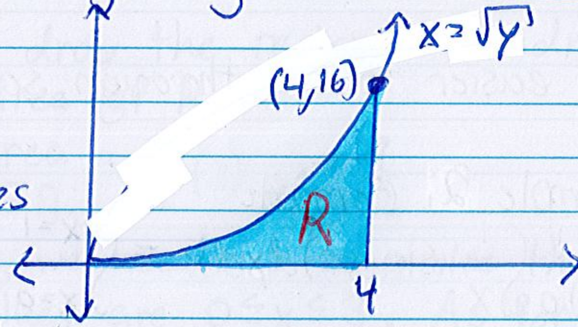
$$= -\frac{1}{2} \cos(1) + \frac{1}{2} \cos(0)$$

$$= \frac{1}{2} - \frac{1}{2} \cos(1)$$

ⓑ $\int_0^{16}\int_{\sqrt{y}}^{4} \sqrt{x^3+1}\,dx\,dy = \int_{y=0}^{y=16}\int_{x=\sqrt{y}}^{x=4} \sqrt{x^3+1}\,dx\,dy$

So the region we are integrating over consist of

$0 \le y \le 16,\ \sqrt{y} \le x \le 4$



So the x-values are $0 \le x \le 4$. The y-values we see the top function is $x=\sqrt{y}$ (or $y=x^2$) and the bottom function is x-axis (or $y=0$). So $0 \le y \le x^2$. Hence

$$\int_{y=0}^{y=16}\int_{x=\sqrt{y}}^{x=4} \sqrt{x^3+1}\,dx\,dy = \int_{x=0}^{x=4}\int_{y=0}^{y=x^2} \sqrt{x^3+1}\,dy\,dx$$

$$= \int_{x=0}^{x=4}\left(\sqrt{x^3+1}\cdot y\Big]_{y=0}^{y=x^2}\right)dx$$

$$= \int_{x=0}^{x=4} x^2\sqrt{x^3+1}\,dx$$

$$\frac{u=x^3+1}{du=3x^2dx} \quad \int u^{1/2}\,\frac{du}{3}$$

$du/3 = x^2dx$

$$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2}$$

$$= \frac{2}{9}(x^3+1)^{3/2}\Big]_{x=0}^{x=4}$$

$$= \frac{2}{9}(4^3+1)^{3/2} - \frac{2}{9}(0^3+1)^{3/2}$$

$$= \frac{2}{9}\cdot(65)^{3/2} - \frac{2}{9}$$