Lesson 2: Integration By Substitution

Definition: For $f(x)$ defined on $[a,b]$, the average value of $f(x)$ on $[a,b]$ is:

$$f_{AVE}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

From the Problem Set (posted online),

Example 1: Find the average value of $f(x) = 6x^2+2$ over $[1,3]$.

$$f_{AVE}(x) = \frac{1}{3-1}\int_1^3 (6x^2+2)\,dx = \frac{1}{2}\int_1^3 (6x^2+2)\,dx$$

$$= \int_1^3 (3x^2+1)\,dx = \left(\frac{3x^3}{3}+x\right)\Big]_1^3 = (x^3+x)\Big]_1^3$$

$$= (3^3+3)-(1^3+1) = 28$$

Example 2: Find the average value of $f(x) = \frac{2x}{x^2+1}$ over $[0,5]$.

$$f_{AVE}(x) = \frac{1}{5-0}\int_0^5 \frac{2x}{x^2+1}\,dx \quad \frac{u = x^2+1}{du = 2x\,dx} \quad \frac{1}{5}\int \frac{du}{u}$$

$$= \frac{1}{5}\ln|u| = \frac{1}{5}\ln|x^2+1|\Big]_0^5 = \frac{1}{5}\ln|5^2+1| - \cancel{\frac{1}{5}\ln|0^2+1|}^{\,0}$$

$$= \frac{1}{5}\ln(26)$$

Example 3: After t months on the job, a postal clerk can sort $Q(t) = 700-400e^{-0.5t}$ letters per hour. What is the average rate at which the clerk sorts mail during the first 3 months on the job?

$$Q_{AVE}(t) = \frac{1}{3-0}\int_0^3 (700-400e^{-0.5t})\,dt$$

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$$= \int_0^3 \frac{700}{3}\,dt - \int_0^3 \frac{400}{3} e^{-0.5t}\,dt$$

$u = -0.5t$

$du = -0.5\,dt$

$-2\,du = dt$

$$= \frac{700}{3}t \Big]_0^3 - \frac{400}{3}\int e^u \cdot (-2)\,du$$

$$= \frac{700}{3}t \Big]_0^3 + \frac{800}{3}\int e^u\,du = \frac{700}{3}t\Big]_0^3 + \frac{800}{3}e^u$$

$$= \frac{700}{3}t\Big]_0^3 + \frac{800}{3}e^{-0.5t}\Big]_0^3$$

$$= \frac{700}{3}\cdot 3 - \frac{700}{3}(0) + \frac{800}{3}e^{-1.5} - \frac{800}{3}e^0$$

$$= \frac{2100-800}{3} + \frac{800}{3}e^{-1.5} = \frac{1300}{3} + \frac{800}{3}e^{-1.5}$$

$\approx \$492.83$

HW 2.3: A certain plant grows at the rate $H'(t) = \frac{1}{\sqrt[3]{8t+3}}$ inches per day, t days after it was planted. How many inches will the height of the plant change on the third day?

$$\int_2^3 \frac{dt}{\sqrt[3]{8t+3}}$$

$u = 8t+3$

$du = 8\,dt$

$\frac{du}{8} = dt$

$$\int \frac{1}{u^{1/3}} \cdot \frac{du}{8} = \frac{1}{8}\int u^{-1/3}\,du$$

$$= \frac{1}{8}\cdot\frac{3}{2}u^{2/3} = \frac{3}{16}(8t+3)^{2/3}\Big]_2^3$$

$$= \frac{3}{16}(27)^{2/3} - \frac{3}{16}(19)^{2/3} \approx 0.352$$

Example 4: Suppose as a particle slows down, its velocity is:

$$v(t) = 2e^{1-t} - 1 \quad \text{cm/s}$$

If the particle starts slowing down at time $t=0$ seconds, find the distance it takes for the particle to stop.

Assume $s(0)=0$. First find when the particle stops by solving

$v(t)=0 \Rightarrow 2e^{1-t}-1=0$

$$2e^{1-t}=1$$
$$e^{1-t}=1/2$$
$$1-t=\ln(1/2)$$
$$t=-\ln(1/2)+1$$

Now find $s(t)$.

$$s(t)=\int v(t)\,dt=\int(2e^{1-t}-1)\,dt=2\int e^{1-t}\,dt-\int dt$$

$u=1-t$

$du=-dt$

$-du=dt$

$$=-2\int e^u\,du-t+C=-2e^u-t+C=-2e^{1-t}-t+C$$

Find C (with $s(0)=0$)

$$0=s(0)=-2e-0+C$$
$$2e=C$$

So $s(t)=-2e^{1-t}-t+2e$

The distance when the particle stops is

$$s(1-\ln(1/2))=-2\exp[1-1+\ln(1/2)]-(1-\ln(1/2))+2e$$
$$=-2\left(\frac{1}{2}\right)-1+\ln\left(\frac{1}{2}\right)+2e$$
$$=2e-2+\ln(1/2)\approx 4.81\text{ cm}$$