Lesson 30: Systems of Equations, Gaussian Elimination

Recall an equation is said to be linear if no two variables are multiplied together.

ex. $5x + 2y - z = 3$ is linear
$3xy - z = 1$ is NOT

A system of linear equations is a group of 2 or more equations with the same set of unknown variables.

We have been taught that there are 3 ways to solve these systems:

① Substitution Method
② Elimination Method
③ Graphical Method

We also saw that there are 3 different types of solutions to a given system:

① Unique Solution
② No solution
③ Infinitely many solutions

A system is consistent if there a solution.

- If we have a unique solution, we have a consisent independent system.
- If we have infinitely many solutions, we have a consistent dependent system.

A system is inconsistent if there is no solution.

Example 1: Solve the given system. Classify each system as consistent independent, consistent dependent, or inconsisent.

ⓐ $\begin{cases} x + 21y = 35 & ① \\ x = 14 & ② \end{cases}$

Plug $x=14$ into ①.
$$14+21y=35$$
$$21y=21$$
$$y=1$$

Hence $(14,1)$ is our solution.

Since we have 1 solution, the system is consistent independent

ⓑ $\begin{cases} 2x+2y=3 & ① \\ x+y=3 & ② \end{cases}$

Multiply ② by -2.
$$-2x-2y=-6 \quad ②$$

Add ① and ②.
$$\begin{array}{rl} & 2x+2y=3 \\ + & -2x-2y=-6 \\ \hline & 0+0=-3 \end{array}$$
$0=-3 \Rightarrow$ Impossible $\Rightarrow$ No Solution

Hence this system is inconsistent.

If we have 2 equations, it isn't to bad to solve for a solution. But when we have 3 or more equations, things can go arwy pretty quickly.

To make things easier, we introduced the idea of augmented matrices.

Definition: $\boxed{2\times 2 \text{ case}}$ Given a system of equations,
$$\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$$
it's augmented matrix is
$$\left[\begin{array}{cc|c} a & b & c \\ d & e & f \end{array}\right]$$
where each column corresponds to the coefficients of x, and y followed by what each equation is equal to.

$\boxed{3 \times 3}$ Given a system of equations,

$$\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases}$$

it's augmented matrix is

$$\left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right]$$

where each column corresponds to the coefficients of x, y, and z followed by what each equation is equal to.

Example 2: Write the augmented matrix that represents the following system of equations.

$$\begin{cases} -2x - y = 1 \\ 2x - 4y - 5z = -2 \\ -x + 2y + 4z = -4 \end{cases}$$

Note when there is a variable missing like in the first equation, add a $0z$ term.

$$\begin{cases} -2x - y + 0z = 1 \\ 2x - 4y - 5z = -2 \\ -x + 2y + 4z = -4 \end{cases}$$

Now that each equation has a x, y, and z, it's augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 1 \\ 2 & -4 & -5 & -2 \\ -1 & 2 & 4 & -4 \end{array}\right]$$

But how do we solve these systems with augmented matrices? Gaussian elimination

Goal of this method is to obtain a matrix of the form

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

where * are just #s. Such a matrix is said to be in row echelon form.

Note that we may not always get all 1's on the diagonal.

- For example, $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$. In this case, we say z is a free variable. The free variable corresponds to the column without the leading 1.

  When a situation like this arises, we set $z=t$ and then solve for x and y in terms of $z=t$.

- Similarly, we can do the same for the case $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ where y is the free variable.

- And lastly, when $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ both y and z are free variables. In this case, let $y=t$ and $z=u$.

<u>Gaussian Elimination</u>

1. Get a 1 in the top left entry.
2. Make every entry below this 1 a 0.
3. Go to the next row and find the first nonzero entry. Make this a 1.
4. Make every entry below this 1 a 0.
5. Repeat this process until you run out of rows.
6. Translate the matrix back into equations to solve.

Now we need to know what are the valid operations to make this happen. These are referred to as the elementary row operations.

Elementary Row Operations
1. Switch any 2 rows
2. Multiply a row by a nonzero #.
3. Add a constant multiple of one row to another.

Example 3: Transform the given augmented matrix into row echelon form.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$\xrightarrow{2R_1-R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2-R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example 4: Use Gaussian Elimination to solve the system of equations.

$$\begin{cases} -x + y - z = 3 \\ 3x + y - z = -1 \\ 2x - y - 2z = -1 \end{cases}$$

First rewrite the system as an augmented matrix. Then apply the Gaussian Elimination.

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 3 \\ 3 & 1 & -1 & -1 \\ 2 & -1 & -2 & -1 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 3 & 1 & -1 & -1 \\ 2 & -1 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{3R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 0 & -4 & 4 & -8 \\ 2 & -1 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{2R_1 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 0 & -4 & 4 & -8 \\ 0 & -1 & 4 & -5 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 4 & -5 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Hence
$$\begin{cases} x - y + z = 3 & (1) \\ y - z = 2 & (2) \\ z = -1 & (3) \end{cases}$$

Plug (3) in (2),

$$y - z = 2$$
$$y + 1 = 2$$
$$y = 1$$

Plug $y=1$ and $z=-1$ into (1),

$$x - y + z = 3$$
$$x - 1 - 1 = 3$$
$$x - 2 = 3$$
$$x = 5$$

Hence the solution is $(5, 1, -1)$.

Example 5: Solve the given system using Gaussian elimination. Classify each system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} -8x + y - 6z = 5 \\ x + 4z = -6 \\ -6x + y + 2z = -7 \end{cases}$$

First rewrite the system as an augmented matrix. Then apply the Gaussian Elimination.

$$\left[\begin{array}{ccc|c} -8 & 1 & -6 & 5 \\ 1 & 0 & 4 & -6 \\ -6 & 1 & 2 & -7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -6 \\ -8 & 1 & -6 & 5 \\ -6 & 1 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{8R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -6 \\ 0 & 1 & 26 & -43 \\ -6 & 1 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{6R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -6 \\ 0 & 1 & 26 & -43 \\ 0 & 1 & 26 & -43 \end{array}\right]$$

$$\xrightarrow{R_2 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -6 \\ 0 & 1 & 26 & -43 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So 0 row $\Rightarrow$ Consistent Dependent

If the system is consistent dependent, let t be the free parameter and express the other variables in terms of t.

Let $z = t$

$$\begin{cases} x + 4t = -6 \\ y + 26t = -43 \end{cases} \longrightarrow \begin{cases} x = -4t - 6 \\ y = -26t - 43 \\ z = t \end{cases}$$