Lesson 31: Gauss-Jordan Elimination

Recall Gaussian Elimination: Given a system of equations,

$$\begin{cases} 3x+2y=1 \\ x+y=1 \end{cases}$$

First, write down its associated augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

Now using row operations, we obtain a matrix in row echelon form.

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow{3R_1 - R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Now rewrite the matrix back to equations and solve for a solution.

$$\begin{cases} x+y=1 & ① \\ y=2 & ② \end{cases}$$

Plug ② into ①.

$$x+2=1$$
$$x=-1$$

Hence our solution is $(-1,2)$.

If we really want to, we could continue to perform row operations to get a matrix of the form:

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

where $*$ are arbitrary numbers. These matrices are said to be in reduced row echelon form.

The algorithm for obtaining a matrix in reduced row echelon form is called Gauss-Jordan elimination.

Example 1: Find the reduced row echelon form of the following matrix:

(a) $\left[\begin{array}{ccc|c} -2 & 1 & 3 & 32 \\ 1 & -1 & 0 & -5 \\ -2 & 0 & 2 & 26 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ -2 & 1 & 3 & 32 \\ -2 & 0 & 2 & 26 \end{array}\right]$

$\xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & -1 & 3 & 22 \\ -2 & 0 & 2 & 26 \end{array}\right]$

$\xrightarrow{2R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & -1 & 3 & 22 \\ 0 & -2 & 2 & 16 \end{array}\right]$

$\xrightarrow[-\frac{1}{2}R_3 \to R_3]{-R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & 1 & -3 & -22 \\ 0 & 1 & -1 & -8 \end{array}\right]$

$\xrightarrow{R_2 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & 1 & -3 & -22 \\ 0 & 0 & -2 & -14 \end{array}\right]$

$\xrightarrow{-\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & 1 & -3 & -22 \\ 0 & 0 & 1 & 7 \end{array}\right]$

$\xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -27 \\ 0 & 1 & -3 & -22 \\ 0 & 0 & 1 & 7 \end{array}\right]$

$\xrightarrow{R_1 + 3R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & -3 & -22 \\ 0 & 0 & 1 & 7 \end{array}\right]$

$\xrightarrow{R_2 + 3R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 7 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ -2 & 2 & -4 & -20 \\ -5 & 2 & -2 & -43 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ 0 & -8 & -6 & 46 \\ -5 & 2 & -2 & -43 \end{array}\right]$

$\xrightarrow{5R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -5 & -1 & 33 \\ 0 & -8 & -6 & 46 \\ 0 & -23 & -7 & 122 \end{array}\right]$

$$\xrightarrow{-3R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&-5&-1&33\\0&-8&-6&46\\0&1&11&-16\end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|c}1&-5&-1&33\\0&1&11&-16\\0&-8&-6&46\end{array}\right]$$

$$\xrightarrow{8R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&-5&-1&33\\0&1&11&-16\\0&0&82&-82\end{array}\right]$$

$$\xrightarrow{\frac{1}{82}R_3\to R_3}\left[\begin{array}{ccc|c}1&-5&-1&33\\0&1&11&-16\\0&0&1&-1\end{array}\right]$$

$$\xrightarrow{R_1+5R_2\to R_1}\left[\begin{array}{ccc|c}1&0&54&-47\\0&1&11&-16\\0&0&1&-1\end{array}\right]$$

$$\xrightarrow{R_2-11R_3\to R_2}\left[\begin{array}{ccc|c}1&0&54&-47\\0&1&0&-5\\0&0&1&-1\end{array}\right]$$

$$\xrightarrow{R_1-54R_3\to R_1}\left[\begin{array}{ccc|c}1&0&0&7\\0&1&0&-5\\0&0&1&-1\end{array}\right]$$

Example 2: Find all solutions to the following system:

ⓐ $\begin{cases} 2x+2y+2z=5\\ 3x+y+5z=13\\ x+2z=4\end{cases}$

$$\left[\begin{array}{ccc|c}2&2&2&5\\3&1&5&13\\1&0&2&4\end{array}\right]\xrightarrow{R_1\leftrightarrow R_3}\left[\begin{array}{ccc|c}1&0&2&4\\3&1&5&13\\2&2&2&5\end{array}\right]$$

$$\xrightarrow{3R_1-R_2\to R_2}\left[\begin{array}{ccc|c}1&0&2&4\\0&-1&1&-1\\2&2&2&5\end{array}\right]$$

$$\xrightarrow{2R_1-R_3\to R_3}\left[\begin{array}{ccc|c}1&0&2&4\\0&-1&1&-1\\0&-2&2&3\end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Translate back to equations,

$$\begin{cases} x + 2z = 4 \\ y - z = 1 \\ 0 = 1 \end{cases} \Rightarrow \text{can never happen} \Rightarrow \text{No Solution}$$

ⓑ $\begin{cases} 2x + 5y + 4z = 3 \\ 2x + 6y + 6z = 2 \\ 3x + 10y + 10z = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & 5 & 4 & 3 \\ 2 & 6 & 6 & 2 \\ 3 & 10 & 10 & 3 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & 5 & 4 & 3 \\ 1 & 3 & 3 & 1 \\ 3 & 10 & 10 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 5 & 4 & 3 \\ 3 & 10 & 10 & 3 \end{array}\right]$$

$$\xrightarrow{2R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 3 & 10 & 10 & 3 \end{array}\right]$$

$$\xrightarrow{3R_1 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 + 3R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Hence our solution is $(1, 1, -1)$.