Lesson 32: Matrix Operations

A $m \times n$ matrix consist of m rows horizontally and the n columns vertically.

ex. $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

2×2 matrix, 2×1 matrix, 2×3 matrix

Matrix Addition/Subtraction: Assuming both matrices have the same dimension, we add/subtract them "component-wise"

ex. $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

Example 1: Perform the following matrix operations:

ⓐ $$\begin{bmatrix} 6 & -3 & 2 \\ -2 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 4 & -5 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 6+2 & -3+1 & 2+0 \\ -2+4 & 4+(-5) & 1+7 \end{bmatrix} = \begin{bmatrix} 8 & -2 & 2 \\ 2 & -1 & 8 \end{bmatrix}$$

ⓑ $$\begin{bmatrix} 1 & 8 \\ -3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 4 & -5 \end{bmatrix}$$
$$= \begin{bmatrix} 1-2 & 8-1 \\ -3-4 & 4-(-5) \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ -7 & 9 \end{bmatrix}$$

ⓒ $$\begin{bmatrix} 2 & -10 \\ -5 & 7 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}$$

2×2 $\quad$ 3×2

Since the dimensions don't match, we can't add them.

Scalar Multiplication of Matrices: We can multiply a matrix of any dimension by a number (or scalar). Again we do it "component-wise"

ex. $$\alpha \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \\ \alpha g & \alpha h & \alpha i \end{bmatrix}$$

Example 2: Perform the following matrix operations:

ⓐ $3\begin{bmatrix} 5 & -3 & 1 \\ 2 & 4 & 9 \end{bmatrix}$

$= \begin{bmatrix} 3(5) & 3(-3) & 3(1) \\ 3(2) & 3(4) & 3(9) \end{bmatrix} = \begin{bmatrix} 15 & -9 & 3 \\ 6 & 12 & 27 \end{bmatrix}$

ⓑ $(-1)\begin{bmatrix} 0 & 5 \\ -2 & -3 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} (-1)(0) & (-1)(5) \\ (-1)(-2) & (-1)(-3) \\ (-1)(0) & (-1)(4) \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 2 & 3 \\ 0 & -4 \end{bmatrix}$

ⓒ $0\begin{bmatrix} 2 & 10 \\ 3 & 15 \end{bmatrix}$

$= \begin{bmatrix} (0)(2) & (0)(10) \\ (0)(3) & (0)(15) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Example 3: Perform the following matrix operations:

$$2\begin{bmatrix} 1 & 0 & 2 \\ 4 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 & 1 \\ 5 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1) & 2(0) & 2(2) \\ 2(4) & 2(1) & 2(0) \\ 2(3) & 2(2) & 2(0) \end{bmatrix} + \begin{bmatrix} 3(0) & 3(0) & 3(1) \\ 3(5) & 3(1) & 3(-2) \\ 3(1) & 3(1) & 3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 8 & 2 & 0 \\ 6 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 15 & 3 & -6 \\ 3 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 0+0 & 4+3 \\ 8+15 & 2+3 & 0+(-6) \\ 6+3 & 4+3 & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 7 \\ 23 & 5 & -6 \\ 9 & 7 & 3 \end{bmatrix}$$

Multiplying Matrices with Matrices: Let A be a $m \times n$ matrix and B be a $n \times k$ matrix. If

$$\begin{array}{cc} A & B \\ m \times \boxed{n \quad\quad n} \times k \end{array}$$

matches

then AB has dimensions $m \times k$.

Before, showing how to do this multiplication, let's introduce a concept known as "Dot Product"

The "Dot Product" is where we multiply matching components, and then sum them up.

ex. $(1, 2, 3) \cdot (7, 9, 11) = 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 = 58$

ex of multiplying matrices w/ matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (a,b)\cdot(e,g) & (a,b)\cdot(f,h) \\ (c,d)\cdot(e,g) & (c,d)\cdot(f,h) \end{bmatrix}$$

$$= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Again this is possible b/c $2 \times \boxed{2 \quad 2} \times 2$

match

Note the ORDER of matrix multiplication is IMPORTANT!

Example 4: Perform the following matrix operations:

ⓐ $\begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$

$2 \times \boxed{2 \quad\quad 2} \times 2$ $\Rightarrow$ We can multiply.

match

$$\begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} (1,-2)\cdot(2,3) & (1,-2)\cdot(-1,5) \\ (0,-3)\cdot(2,3) & (0,-3)\cdot(-1,5) \end{bmatrix}$$

$$= \begin{bmatrix} 1(2) + (-2)(3) & 1(-1) + (-2)(5) \\ 0(2) + (-3)(3) & 0(-1) + (-3)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 2-6 & -1-10 \\ 0-9 & 0-15 \end{bmatrix} = \begin{bmatrix} -4 & -11 \\ -9 & -15 \end{bmatrix}$$

ⓑ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$2 \times \boxed{3 \quad 3} \times 1$ ⇒ We can multiply.

matches

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} (1,2,3)\cdot(1,0,2) \\ (4,5,6)\cdot(1,0,2) \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+2(0)+3(2) \\ 4(1)+5(0)+6(2) \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$$

Example 5: Find AB and BA, when $A = \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

$1 \times \boxed{3 \quad 3} \times 1$ ⇒ We can multiply. Also our answer is 1×1 matrix.

matches

$$= \begin{bmatrix} (2,-1,4)\cdot(1,5,2) \end{bmatrix}$$
$$= \begin{bmatrix} 2(1)+(-1)(5)+4(2) \end{bmatrix}$$
$$= \begin{bmatrix} 5 \end{bmatrix}$$

$BA = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$

$3 \times \boxed{1 \quad 1} \times 3$ ⇒ We can multiply. Also our answer is 3×3 matrix.

matches

$$BA = \begin{bmatrix} 1(2) & 1(-1) & 1(4) \\ 5(2) & 5(-1) & 5(4) \\ 2(2) & 2(-1) & 2(4) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 4 \\ 10 & -5 & 20 \\ 4 & -2 & 8 \end{bmatrix}$$

So in general $AB \neq BA$.

<u>Example 6</u>: Find A^2, when $A = \begin{bmatrix} -5 & 4 \\ 4 & 4 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} -5 & 4 \\ 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} -5 & 4 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (-5,4)\cdot(-5,4) & (-5,4)\cdot(4,4) \\ (4,4)\cdot(-5,4) & (4,4)\cdot(4,4) \end{bmatrix}$$

$$= \begin{bmatrix} (-5)(-5)+4(4) & (-5)(4)+4(4) \\ 4(-5)+4(4) & 4(4)+4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 41 & -4 \\ -4 & 32 \end{bmatrix}$$