Lesson 33: Inverses of Matrices

Definition: The $n \times n$ identity matrix, denoted by I, is a matrix with 1's on the diagonal and 0's elsewhere.

ex. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Definition: If A and B are square matrices such that
$$AB = BA = I$$
then B is the inverse of A, denoted by A^{-1}.

2x2 Inverses

To find the inverse of a 2×2 matrix there is a convenient formula.

Formula: Given an invertible matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1: Find the inverse of $A = \begin{pmatrix} 2 & 5 \\ 7 & 10 \end{pmatrix}$

By the formula,

$$A^{-1} = \frac{1}{2(10)-5(7)}\begin{bmatrix} 10 & -5 \\ -7 & 2 \end{bmatrix} = \frac{-1}{15}\begin{bmatrix} 10 & -5 \\ -7 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -10/15 & 5/15 \\ 7/15 & -2/15 \end{bmatrix}$$

3x3 Inverses

The process for finding the inverses of 3×3 matrix is a bit more complicated. Sadly there is no formula.

How to find a $n \times n$ inverse

Given a $n \times n$ matrix, A, we do the following:

1. Set up an augmented matrix $[A \mid I]$
2. Perform elementary row operations until you obtain an augmented matrix $[I \mid B]$
3. The matrix B is A^{-1}.

Example 2: Find the inverse of $A = \begin{pmatrix} 3 & 1 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & -4 \end{pmatrix}$

Set up the augmented matrix $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right]$$

now perform row operations to obtain $[I \mid B]$.

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 5 & 1 & 0 & 0 \\ 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{3R_1 - R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & 3 & 0 \\ 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_1 - R_3 \to R_3]{-R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -3 & 0 \\ 0 & -1 & 6 & 0 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 8 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{8}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1/8 & -1/8 & -1/8 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & 9/8 & 1/8 \\ 0 & 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1/8 & -1/8 & -1/8 \end{array}\right]$$

$$R_2 - 2R_3 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & 9/8 & 1/8 \\ 0 & 1 & 0 & 6/8 & -22/8 & 2/8 \\ 0 & 0 & 1 & 1/8 & -1/8 & -1/8 \end{array}\right]$$

Hence $A^{-1} = \begin{pmatrix} -1/8 & 9/8 & 1/8 \\ 6/8 & -22/8 & 2/8 \\ 1/8 & -1/8 & -1/8 \end{pmatrix}$

How will an inverse matrix help us solve a system of equations?

Let $\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases}$

We can translate it to the matrix equation $AX = B$:

$$\underbrace{\begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} d \\ h \\ l \end{bmatrix}}_{B}$$

Let's solve this matrix equation:

$$AX = B$$
$$(A^{-1}A)X = A^{-1}B$$
$$I\,X = A^{-1}B$$
$$X = A^{-1}B$$

Hence if we are given a system of equations, we can write it as $AX = B$. Find A^{-1} and multiply it by B, to get our answer.

Example 3: Solve the system of equations by first finding and then using the inverse of the coefficient matrix

$$\begin{cases} 8x + y = -36 \\ 5x + 8y = 7 \end{cases}$$

Let's translate the system to the matrix equation: $AX=B$

$$\underbrace{\begin{bmatrix} 8 & 1 \\ 5 & 8 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -36 \\ 7 \end{bmatrix}}_{B}$$

Next find A^{-1}. Note you can use the formula

$$A^{-1} = \frac{1}{8(8)-1(5)} \begin{bmatrix} 8 & -1 \\ -5 & 8 \end{bmatrix} = \frac{1}{59} \begin{bmatrix} 8 & -1 \\ -5 & 8 \end{bmatrix}$$

So

$$X = A^{-1}B = \frac{1}{59} \begin{bmatrix} 8 & -1 \\ -5 & 8 \end{bmatrix} \cdot \begin{bmatrix} -36 \\ 7 \end{bmatrix}$$

$$= \frac{1}{59} \begin{bmatrix} 8(-36)-1(7) \\ -5(-36)+8(7) \end{bmatrix}$$

$$= \frac{1}{59} \begin{bmatrix} -295 \\ 236 \end{bmatrix}$$

$$= \begin{bmatrix} -295/59 \\ 236/59 \end{bmatrix}$$

Hence $x = -\frac{295}{59}$, $y = \frac{236}{59}$