Lesson 34: Determinants of Matrices

Last time we mentioned the term "invertible." Let's define that word.

Definition: A matrix is invertible if the determinant is nonzero.

Definition: A matrix is singular if the determinant is zero.

But what is the determinant?

Notation: The determinant of a matrix is written by replacing the brackets/parenthesis with vertical bars.

ex. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2x2 Determinants

Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = ad - bc$.

Note this should be familar since the formula for inverses, in the 2×2 case, has this quantity in the fraction.

Example 1: Compute the determinant of $A = \begin{bmatrix} 8 & 1 \\ 6 & -10 \end{bmatrix}$

By the formula,

$$\det A = 8(-10) - (1)(6) = -80 - 6 = -86$$

3 x 3 Determinants

Sadly there is no simple formula. Instead we find it through a method called cofactor expansion. Before we present this method, we need a couple definitions.

Definition: The i,j minor of the matrix, denoted by M_{ij}, is the determinant that results from deleting the i-th row and the j-th column of the matrix.

Example 2: Let $A = \begin{pmatrix} 6 & 1 & 7 \\ 3 & 2 & 0 \\ 5 & 8 & 4 \end{pmatrix}$. Find the following minors

(a) M_{11}

First delete the 1st row and 1st column.

i.e. $\begin{pmatrix} \cancel{6} & \cancel{1} & \cancel{7} \\ \cancel{3} & 2 & 0 \\ \cancel{5} & 8 & 4 \end{pmatrix}$

So $M_{11} = \begin{vmatrix} 2 & 0 \\ 8 & 4 \end{vmatrix} = 2(4) - 0(8) = 8$

(b) M_{12}

First delete the 1st row and 2nd column.

i.e. $\begin{pmatrix} \cancel{6} & \cancel{1} & \cancel{7} \\ 3 & \cancel{2} & 0 \\ 5 & \cancel{8} & 4 \end{pmatrix}$

So $M_{12} = \begin{vmatrix} 3 & 0 \\ 5 & 4 \end{vmatrix} = 3(4) - 0(5) = 12$

(c) M_{13}

First delete the 1st row and 3rd column

i.e. $\begin{pmatrix} \cancel{6} & \cancel{1} & \cancel{7} \\ 3 & 2 & \cancel{0} \\ 5 & 8 & \cancel{4} \end{pmatrix}$

So $M_{13} = \begin{vmatrix} 3 & 2 \\ 5 & 8 \end{vmatrix} = 3(8) - 2(5) = 24 - 10 = 14$

Definition: The i,j cofactor of the matrix is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Example 3: (Continuing with Example 2) Let $A = \begin{pmatrix} 6 & 1 & 7 \\ 3 & 2 & 0 \\ 5 & 8 & 4 \end{pmatrix}$.

Find the following cofactors.

ⓐ C_{11}

By the formula $C_{11} = (-1)^{1+1} M_{11} = M_{11}$

From Ex 2a, $M_{11} = 8$.

Hence $C_{11} = M_{11} = 8$

ⓑ C_{12}

By the formula $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$

From Ex 2b, $M_{12} = 12$

Hence $C_{12} = -M_{12} = -12$

ⓒ C_{13}

By the formula $C_{13} = (-1)^{1+3} M_{13} = M_{13}$

From Ex 2c, $M_{13} = 14$

Hence $C_{13} = M_{13} = 14$

Definition: Let A be a $n \times n$ matrix, the cofactor expansion along the i-th row is defined with the following formula

$$\det A = \sum_{j=1}^{n} A_{ij} C_{ij}$$

So basically fix a row and calculate.

Example 4: Calculate the determinant of $A = \begin{pmatrix} 6 & 1 & 7 \\ 3 & 2 & 0 \\ 5 & 8 & 4 \end{pmatrix}$

Let's fix the 1st row. So

$$\det A = \sum_{j=1}^{3} A_{1j} C_{1j} = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13}$$

Note that in Example 3, we calculated C_{11}, C_{12}, and C_{13} which were

$$C_{11} = 8 \qquad C_{12} = -12 \qquad C_{13} = 14$$

From the matrix we can find A_{11}, A_{12}, and A_{13}

$$A_{11} = 6 \qquad A_{12} = 1 \qquad C_{13} = 7$$

Putting it all together

$$\det A = 6(8) + 1(-12) + 7(14) = 134$$

An application of this concept is as follows:

Example 5: Solve for x. $\begin{vmatrix} x+1 & 1 \\ 2 & x \end{vmatrix} = 0$

$$\begin{vmatrix} x+1 & 1 \\ 2 & x \end{vmatrix} = x(x+1) - 1(2) = 0$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2, 1$$