Lesson 35: Eigenvalues + Eigenvectors 2×2 Case

Definition: A scalar value λ is called an eigenvalue when there is a vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. These vector(s) $\vec{x}$ are called eigenvectors.

Example 1: Which of the following are eigenvalues and eigenvectors of $A = \begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix}$?

(a) $\lambda = 2$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

LHS: $A\vec{x} = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+4 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

RHS: $\lambda\vec{x} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Since LHS = RHS, $\lambda = 2$ is an eigenvalue and $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A.

(b) $\lambda = 3$, $\vec{x} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

LHS: $A\vec{x} = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2(-4)+4(-1) \\ 1(-4)+1(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

RHS: $\lambda\vec{x} = 3\begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ -3 \end{bmatrix}$

Since LHS ≠ RHS, either $\lambda = 3$ isn't an eigenvalue or $\vec{x} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$ isn't an eigenvector of A.

But how do you find them, in general?

Finding Eigenvalues (i.e. Finding λ)

From the definition, we are given by

$$A\vec{x} = \lambda\vec{x}$$
$$0 = \lambda\vec{x} - A\vec{x}$$
$$0 = \lambda I\vec{x} - A\vec{x}$$
$$0 = \vec{x}(\lambda I - A)$$

So $\lambda I - A = 0$ or $\vec{x} = 0$.

If $\vec{x} = 0$ then we have the trivial solution.

But we want λ. So we are going to focus on $\lambda I - A = 0$.

The issue with this result is that $\lambda I - A = 0$ is a matrix equation, meaning to solve it we need to solve for 4 entries in the 2×2 case and 9 entries in the 3×3 case. To avoid this we can simply look at it's characteristic equation.

Definition: The characteristic equation is a polynomial of degree n that arises from

$$\det(\lambda I - A) = 0$$

Example 2: Find the eigenvalue of the matrices

ⓐ $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 4 \\ 2 & \lambda-8 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda-1)(\lambda-8) - 2(4) = 0$$
$$\lambda^2 - \lambda - 8\lambda + 8 - 8 = 0$$
$$\lambda^2 - 9\lambda = 0$$
$$\lambda(\lambda - 9) = 0$$
$$\lambda = 0,\ 9$$

Hence $\lambda = 0$ and $\lambda = 9$ are eigenvalues of A.

ⓑ $B = \begin{bmatrix} 4 & -2 \\ 5 & 11 \end{bmatrix}$

$$\lambda I - B = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 5 & 11 \end{bmatrix} = \begin{bmatrix} \lambda - 4 & 2 \\ -5 & \lambda - 11 \end{bmatrix}$$

$$\det(\lambda I - B) = (\lambda - 4)(\lambda - 11) - 2(-5) = 0$$
$$\lambda^2 - 4\lambda - 11\lambda + 44 + 10 = 0$$
$$\lambda^2 - 15\lambda + 54 = 0$$
$$(\lambda - 6)(\lambda - 9) = 0$$
$$\lambda = 6, 9$$

Hence $\lambda = 6$ and $\lambda = 9$ are eigenvalues of B.

Finding Eigenvectors

Recall eigenvectors are those $\vec{x}$ such that $A\vec{x} = \lambda \vec{x}$,

Example 3: Which of the following are eigenvectors of
$$A = \begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}?$$

To answer this question multiply A and the given vector. If the result is a multiple of the given vector, then it's a eigenvector of A.

ⓐ $\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -7(3) + 6(3) \\ -4(3) + 4(3) \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Obviously $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ isn't a multiple of $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Hence it's not an eigen vector.

(b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} -7 & 6 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7(2)+6(1) \\ -4(2)+4(1) \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$$

Note that we can factor a -4 and get $A\vec{x} = -4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A.

But how do you find them, in general?
1. Find the eigenvalues via $\det(\lambda I - A) = 0$
2. For each λ, set up the augment matrix $[\lambda I - A \mid 0]$.
   - ↳ If you want you could reduce the matrix.
3. Convert the augmented matrix to system of equations
4. Determine the value for $\vec{x}$.

Example 4: Find the eigenvectors of the matrices.

(a) $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

From Ex 2, we found A's eigenvalues are $\lambda = 0, 9$.

- $\lambda = 0$: $\lambda I - A = 0 \Rightarrow \left[\begin{array}{cc|c} \lambda-1 & 4 & 0 \\ 2 & \lambda-8 & 0 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 2 & -8 & 0 \end{array}\right] \Rightarrow \begin{cases} -x+4y=0 \\ 2x-8y=0 \end{cases} \Rightarrow \{ x = 4y$$

A potential solution is $x=4$ and $y=1$. Hence an eigenvector for $\lambda=0$ is

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- $\lambda = 9$: $\lambda I - A = 0 \Rightarrow \left[\begin{array}{cc|c} \lambda-1 & 4 & 0 \\ 2 & \lambda-8 & 0 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{cc|c} 9-1 & 4 & 0 \\ 2 & 9-8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 8 & 4 & 0 \\ 2 & 1 & 0 \end{array}\right] \Rightarrow \begin{cases} 8x+4y=0 \\ 2x+y=0 \end{cases}$$

$\Rightarrow y = -2x$

A potential solution is $x = -1$ and $y = 2$. Hence an eigenvector for $\lambda = 9$ is

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 4 & -2 \\ 5 & 11 \end{bmatrix}$

From Ex 2, we found B's eigenvalues are $\lambda = 6, 9$

- $\underline{\lambda = 6}$: $\lambda I - B = 0 \Rightarrow \left[\begin{array}{cc|c} \lambda-4 & 2 & 0 \\ -5 & \lambda-11 & 0 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{cc|c} 6-4 & 2 & 0 \\ -5 & 6-11 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ -5 & -5 & 0 \end{array}\right] \Rightarrow \begin{cases} 2x + 2y = 0 \\ -5x - 5y = 0 \end{cases}$$

$\Rightarrow y = -x$

A potential solution is $x = 1$ and $y = -1$. Hence an eigenvecter for $\lambda = 6$ is

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- $\underline{\lambda = 9}$: $\lambda I - B = 0 \Rightarrow \left[\begin{array}{cc|c} \lambda-4 & 2 & 0 \\ -5 & \lambda-11 & 0 \end{array}\right]$

$$\Rightarrow \left[\begin{array}{cc|c} 9-4 & 2 & 0 \\ -5 & 9-11 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 5 & 2 & 0 \\ -5 & -2 & 0 \end{array}\right] \Rightarrow \begin{cases} 5x + 2y = 0 \\ -5x - 2y = 0 \end{cases}$$

$\Rightarrow 5x + 2y = 0$

A potential solution is $x = -2$ and $y = 5$. Hence an eigen vector for $\lambda = 9$ is

$$\vec{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$