Lesson 36: Eigenvalues + Eigenvectors 3×3 Case

Recall from Last Time, how we find eigenvalues and eigenvectors:

① Find the eigenvalues via $\det(\lambda I - A) = 0$

② For each λ, set up the augmented matrix $[\lambda I - A \mid 0]$

↳ In the 3x3 case, it is highly encourage to row reduce the matrix.

③ Convert the augmented matrix to a system of equations.

④ Determine the value for $\vec{x}$.

The main difference in finding the eigenvalues and eigenvectors in the 2x2 and 3x3 case is how we compute

$$\det(\lambda I - A) = 0$$

Recall from Lesson 34,

<u>Definition</u>: Let A be a nxn matrix, the cofactor expansion along the i-th row is defined with the following formula:

$$\det(A) = \sum_{j=1}^{n} A_{ij} C_{ij}$$

<u>Definition</u>: The i,j cofactor of the matrix is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

<u>Definition</u>: The i,j minor of the matrix, denoted by M_{ij}, is the determinant that results from deleting the i-th row and the j-th column of the matrix.

<u>Example 1</u>: Find the eigenvalues and eigenvectors for

$$\begin{bmatrix} 2 & -8 & -7 \\ 0 & -9 & -5 \\ 0 & 7 & 3 \end{bmatrix}$$

To find the eigenvalues we need to take the determinant of

$$\lambda I - A = \begin{bmatrix} \lambda-2 & 8 & 7 \\ 0 & \lambda+9 & 5 \\ 0 & -7 & \lambda-3 \end{bmatrix}$$

To calculate $\det(\lambda-IA)$ we need to calculate M_{11}, M_{12}, M_{13} and C_{11}, C_{12}, C_{13}.

$$M_{11} = \begin{vmatrix} \lambda+9 & 5 \\ -7 & \lambda-3 \end{vmatrix} = (\lambda+9)(\lambda-3) - (5)(-7)$$
$$= \lambda^2 + 9\lambda - 3\lambda - 27 + 35$$
$$= \lambda^2 + 6\lambda + 8$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = \lambda^2 + 6\lambda + 8$$

$$M_{12} = \begin{vmatrix} 0 & 5 \\ 0 & \lambda-3 \end{vmatrix} = 0(\lambda-3) - 0(5) = 0$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = 0$$

$$M_{13} = \begin{vmatrix} 0 & \lambda+9 \\ 0 & -7 \end{vmatrix} = 0(7) - 0(\lambda+9) = 0$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13} = 0$$

So $\det(\lambda-IA) = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13}$
$$= (\lambda-2)(\lambda^2+6\lambda+8) + (8)(0) + (7)(0)$$
$$= (\lambda-2)(\lambda^2+6\lambda+8) = 0$$

Luckily $\det(\lambda-IA)$ is partly factor. So let's finish it off.
$$(\lambda-2)(\lambda+2)(\lambda+4) = 0$$
Hence the eigenvalues of A is $\lambda = -2, 2, -4$.

Next we need to find the eigenvector(s) for each λ.
Recall:
$$\lambda-IA = \begin{bmatrix} \lambda-2 & 8 & 7 \\ 0 & \lambda+9 & 5 \\ 0 & -7 & \lambda-3 \end{bmatrix}$$

• $\lambda = -2$: $\left[\begin{array}{ccc|c} -2-2 & 8 & 7 & 0 \\ 0 & -2+9 & 5 & 0 \\ 0 & -7 & -2-3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -4 & 8 & 7 & 0 \\ 0 & 7 & 5 & 0 \\ 0 & -7 & -5 & 0 \end{array}\right]$

$$R_2+R_3 \to R_3 \left[\begin{array}{ccc|c} -4 & 8 & 7 & 0 \\ 0 & 7 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{7R_1-8R_2 \to R_1} \left[\begin{array}{ccc|c} -28 & 0 & -5 & 0 \\ 0 & 7 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} -28x - 5z = 0 \\ 7y + 5z = 0 \end{cases}$$

To determine a solution of this system, set one variable equal to 1 (or any # of your choosing) and find the rest of the variables.

Let $x = 1$. Then
$$-28x - 5z = 0$$
$$-28 - 5z = 0$$
$$-28 = 5z$$
$$z = -28/5$$

Plug $z = -28/5$ into
$$7y + 5z = 0$$
$$7y + 5(-28/5) = 0$$
$$7y - 28 = 0$$
$$7y = 28$$
$$y = 4$$

Hence the eigenvector for $\lambda = -2$ is $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ -28/5 \end{bmatrix}$

• $\underline{\lambda = -4}$: $\left[\begin{array}{ccc|c} -4-2 & 8 & 7 & 0 \\ 0 & -4+9 & 5 & 0 \\ 0 & -7 & -4-3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -6 & 8 & 7 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -7 & -7 & 0 \end{array}\right]$

$$\xrightarrow[-\frac{1}{7}R_3 \to R_3]{\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c} -6 & 8 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_1-8R_2 \to R_1]{R_2-R_3 \to R_3} \left[\begin{array}{ccc|c} -6 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} -6x - z = 0 \\ y + z = 0 \end{cases}$$

Again to determine a solution of this system, set one variable equal to 1 and find the rest of the variables.

Let $x=1$. Then $-6x - z = 0$

$$-6 - z = 0$$
$$z = -6.$$

Plug $z=-6$ into $y+z=0$

$$y-6=0$$
$$y=6$$

Hence the eigenvector for $\lambda=-4$ is $\vec{x} = \begin{bmatrix} 1 \\ 6 \\ -6 \end{bmatrix}$

• $\underline{\lambda=2}$: $\left[\begin{array}{ccc|c} 2-2 & 8 & 7 & 0 \\ 0 & 2+9 & 5 & 0 \\ 0 & -7 & 2-3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 8 & 7 & 0 \\ 0 & 11 & 5 & 0 \\ 0 & -7 & -1 & 0 \end{array}\right]$

$\Rightarrow \begin{cases} 8y+7z=0 \\ 11y+5z=0 \\ -7y-z=0 \end{cases}$ The only way all 3 eqns are true is if $y=0$ and $z=0$.

Note none of these equations have x. So x is anything. So let $x=1$. Hence the eigenvector for $\lambda=2$ is $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Before we do another example let's practice finding solutions of a polynomial via grouping and synthetic division.

Example 2: Find all the solutions to the equations:

ⓐ $x^3+4x^2-36x-144=0$

In this example, we will find the solutions via grouping

$$\underbrace{x^3+4x^2}\underbrace{-36x-144}=0$$

$$x^2(x+4)-36(x+4)=0$$
$$(x^2-36)(x+4)=0$$

$$(x-6)(x+6)(x+4)=0$$

Hence the solutions are $x=-6,-4,6$

(b) $x^3-2x^2-11x+12=0$

In this example, we will find the solutions via synthetic division.

If you want a more detail explanation of this method refer to the Algebra Review document posted online.

Choose a # and check that the equation is indeed 0 at that #.

Let's try with $x=1$.

$$1^3-2(1)^2-11(1)+12 = 1-2-11+12 = 0 \checkmark$$

So we are doing synthetic division with $x=1$.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & \downarrow & 1 & -1 & -12 \\ \hline & 1 & -1 & -12 & 0 \\ & x^2 & x & \text{constant} & \end{array}$$

So $x^3-2x^2-11x+12 = (x-1)(x^2-x-12)=0$

$$(x-1)(x-4)(x+3)=0$$

Hence the solutions are $x=-3,1,4$.