Lesson 3: The Natural Logarithmic Function: Integration

The function $f(t) = 1/t$ is continuous on $(0, \infty)$.

By FTC, $f(t) = 1/t$ has an antiderivative on the interval with endpoints x and 1 when $x > 0$.

Definition: $\ln(x) = \int_1^x \frac{1}{t}\,dt$

But in general, $\ln|x| = \int \frac{dt}{t}$

Example 1: Let $f(x) = \ln(x^5 + 7x + 12)$. Compute $f'(x)$.

$$f'(x) = \frac{1}{x^5+7x+12} \cdot (x^5+7x+12)'$$

$$= \frac{5x^4+7}{x^5+7x+12}$$

Example 2: Evaluate

ⓐ $\int \frac{5x^4+7}{x^5+7x+12}\,dx$ $\quad \frac{u = x^5+7x+12}{du = (5x^4+7)dx}$ $\quad \int \frac{du}{u} = \ln|u| + C$

$$= \ln|5x^4+7| + C$$

ⓑ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\quad \frac{u = \cos x}{du = -\sin x\,dx}$ $\quad \int -\frac{du}{u}$

$-du = \sin x\,dx$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

ⓒ $\int \frac{dx}{x\ln(x^{19})}$ $\quad \frac{u = \ln(x^{19})}{du = \frac{19x^{18}}{x^{19}}dx}$ $\quad \int \frac{1}{u}\cdot\frac{du}{19} = \frac{1}{19}\ln|u| + C$

$du = \frac{19}{x}\,dx$

$\frac{du}{19} = \frac{dx}{x}$

$$= \frac{1}{19}\ln\left|\ln(x^{19})\right| + C$$

Example 3: Solve the IVP for f(x):

$$\frac{dy}{dx} = \frac{\ln\sqrt{x}}{x} \quad \text{where } y=2 \text{ when } x=1$$

$$y = \int \frac{dy}{dx}\,dx = \int \frac{\ln x^{1/2}}{x}\,dx = \frac{1}{2}\int \frac{\ln x}{x}\,dx \qquad \frac{u=\ln x}{du = 1/x\,dx}$$

$$= \frac{1}{2}\int u\,du = \frac{1}{2}\cdot\frac{u^2}{2} + C = \frac{1}{4}(\ln x)^2 + C$$

Now find C with $y(1)=2$.

$$2 = \frac{1}{4}(\ln(1))^2 + C$$

$$2 = C \Rightarrow y = \frac{1}{4}(\ln(x))^2 + 2$$

Example 4: Evaluate

Extra Problem

$$\int_1^2 \frac{1-\cos\theta}{\theta - \sin\theta}\,d\theta \qquad \frac{u = \theta - \sin\theta}{du = 1-\cos\theta\,d\theta} \qquad \int \frac{du}{u} = \ln|u|$$

$$= \ln|\theta - \sin\theta|\Big]_1^2$$

$$= \ln|2-\sin(2)| - \ln|1-\sin(1)|$$

Example 5: Compute the area of the region whose borders are given by

$$y = 2x - \tan(0.3x) \qquad x=1 \qquad x=4 \qquad y=0$$

Extra Problem

$$\int_1^4 (2x - \tan(0.3x))\,dx = \int_1^4 2x\,dx - \underbrace{\int_1^4 \frac{\sin(0.3x)}{\cos(0.3x)}\,dx}$$

$u = \cos(0.3x)$

$du = -\sin(0.3x)\cdot 0.3dx$

$-\frac{du}{0.3} = \sin(0.3x)dx$

$$= \frac{2x^2}{2}\Big]_1^4 - \int \frac{1}{u}\cdot\frac{-du}{0.3}$$

$$= x^2 \Big]_1^4 + \frac{1}{0.3} \ln|u|$$

$$= x^2 \Big]_1^4 + \frac{1}{0.3} \ln|\cos(0.3x)| \Big]_1^4$$

$$= 4^2 - 1^2 + \frac{1}{0.3} \ln|\cos(1.2)| - \frac{1}{0.3} \ln|\cos(0.3)|$$

$$\approx 11.7686$$