Lesson 4: Integration by Parts

Recall the Product Rule

$$(uv)' = u'v + uv'$$

What if we integrate both sides with respect to x.

$$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

Remember $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$. So

$$uv = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

$$uv = \int v\,du + \int u\,dv$$

$$uv - \int v\,du = \int u\,dv$$

Interesting enough is that for many cases an integral will have the form

$$\int u\,du$$

This technique of turning one integral into another is called Integration by Parts.

It's formula is: $\int u\,dv = uv - \int v\,du$

To use this technique choose u to be the one to take derivative or and dv to be integrated.

Example 1: Evaluate

ⓐ $\int x e^{0.2x}\,dx$ $\quad \dfrac{u = x \qquad dv = e^{0.2x}\,dx}{du = dx \qquad v = \frac{1}{0.2}e^{0.2x}}$ (next page)

$= 5xe^{0.2x} - \int 5e^{0.2x}\,dx = 5xe^{0.2x} - \frac{5}{0.2}e^{0.2x} + C$

$= 5xe^{0.2x} - 25e^{0.2x} + C$

ⓑ $\int x \ln x\,dx$ $\quad \frac{u = \ln x \qquad dv = x\,dx}{du = \frac{1}{x}dx \qquad v = x^2/2}$ $\quad \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$

$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2 \ln x}{2} - \frac{1}{2}\cdot\frac{x^2}{2} + C$

$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$

Example 2: Evaluate

ⓐ $\int_0^{\pi/2} x \sin x\,dx$ $\quad \frac{u = x \qquad dv = \sin x\,dx}{du = dx \qquad v = -\cos x}$

$= -x\cos x\Big]_0^{\pi/2} - \int_0^{\pi/2}(-\cos x)\,dx$

$= -x\cos x\Big]_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx$

$= -x\cos x\Big]_0^{\pi/2} + \sin x\Big]_0^{\pi/2}$

$= \underbrace{-\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)}_{0} + \underbrace{0\cos(0)}_{0} + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} - \underbrace{\sin(0)}_{0}$

$= 1$

ⓑ $\int_1^{e^2} x \ln \sqrt[3]{x}\,dx = \int_1^{e^2} x \ln x^{1/3}\,dx = \frac{1}{3}\int_1^{e^2} x \ln x\,dx$

By 1b, $= \frac{1}{3}\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right)\Big]_1^{e^2}$

$= \frac{1}{3}\left(\frac{(e^2)^2 \ln e^2}{2} - \frac{(e^2)^2}{4}\right) - \frac{1}{3}\left(\frac{\cancel{\ln(1)}^{0}}{2} - \frac{1}{4}\right)$

$= \frac{1}{3}\left(\frac{e^4 \cdot \cancel{2}}{\cancel{2}} - \frac{e^4}{4}\right) + \frac{1}{12}$

$= \frac{1}{3}\cdot\frac{3e^4}{4} + \frac{1}{12} = \frac{e^4}{4} + \frac{1}{12}$

HW 4.2: Evaluate

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{x \cdot x^2}{\sqrt{4+x^2}}\,dx \qquad \frac{u = 4+x^2 \iff u-4 = x^2}{du = 2x\,dx \iff \frac{du}{2} = x\,dx}$$

$$= \int \frac{u-4}{u^{1/2}}\,\frac{du}{2} = \frac{1}{2}\int u^{1/2} - 4u^{-1/2}\,du$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{3/2} - 4\cdot\frac{\cancel{2}}{1}u^{1/2}\right) + C$$

$$= \frac{1}{3}(4+x^2)^{3/2} - 4(4+x^2)^{1/2} + C$$