Lesson 6: Differential Equations

Definition: A differential equation is an equation that relates one or more functions and their derivatives.

e.g. ⓐ $y' = ky$ ⓑ $y' = C$ ⓒ $y'' = y$

Definition: The general solution to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Answers are of the form $y = \underline{\qquad\qquad} + C$

Definition: The particular solution is similar to the general solution but it does take initial condition.

i.e. Find the general solution. Then using the initial condition to find C. Plug C back into the general solution and done.

Note: Solving the Initial Value Problem (IVP) is the same to finding the particular solution.

A) Growth & Decay

Recall Lessons 35 and 36 from last semester.

Example 1: Consider the differential equation $\frac{dy}{dx} = ky$ where the proportionally constant $k > 0$.

Idea: Try to get terms w/ y on one-side and x on the other.

$$\frac{dy}{dx} = ky$$

$$\frac{dy}{dx}\,dx = ky\,dx$$

$$dy = ky\,dx$$

$$\frac{1}{y}\,dy = \frac{1}{y}(ky)\,dx$$

$$\frac{1}{y}\,dy = k\,dx$$

Now integrate.

$$\int \frac{1}{y}\,dy = \int k\,dx$$

$$\ln|y| = kx + c$$

$$e^{\ln|y|} = e^{kx+c}$$

$$|y| = e^{kx}e^{c}$$

$$\pm y = e^{kx}e^{c}$$

$$y = \underbrace{\pm e^{c}}_{\text{All of this is a constant... so call it all C.}} e^{kx}$$

$$y = Ce^{kx}$$

In the future, proportionality $\Rightarrow y' = ky$

Also recall that half-life constant is denoted as

$$k = \frac{\ln(1/2)}{\text{half-life}} = \frac{-\ln 2}{\text{half-life}}$$

B) Separation of Variables

The technique used in ex 1 is called Separation of Variable.

Example 2: Find general solution of the differential equation: $\frac{dy}{dx} = \frac{x}{y}$

Solve like we did in example 1.

$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 = x^2 + 2C = x^2 + C$$

$$y = \pm\sqrt{x^2 + C}$$

Example 3: Solve the IVP

$$\frac{dy}{dt} = 2(25-y) \quad , \; y(0) = 40$$

Rewrite: $\frac{dy}{25-y} = 2dt$

$$\underbrace{\int \frac{dy}{25-y}}_{u=25-y} = \int 2dt$$

$u = 25-y$

$du = -dy$

$-du = dy$

$$\int \frac{-du}{u} = \int 2dt$$

$$-\ln|u| = 2t + C$$

$$-\ln|25-y| = 2t + C$$

$$\ln|25-y|^{-1} = 2t + C$$

$$\ln\left|\frac{1}{25-y}\right| = 2t + C$$

$$\exp\left[\ln\left|\frac{1}{25-y}\right|\right] = \exp[2t + C]$$

$$\left|\frac{1}{25-y}\right| = e^{2t}e^{C}$$

$$\pm\frac{1}{25-y} = e^{C}e^{2t}$$

$$\frac{1}{25-y} = \pm e^{C}e^{2t}$$

$$\frac{1}{25-y} = Ce^{2t}$$

$$25-y = \frac{1}{Ce^{2t}} = \frac{1}{C}e^{-2t} = Ce^{-2t}$$

$$y = 25 - Ce^{-2t}$$

Now let's find C w/ $y(0)=40$.

$$40 = 25 - Ce^{0}$$

$$40 = 25 - C$$

$$15 = -C$$

$$C = -15$$

Hence $y = 25 + 15e^{-2t}$

Example 4: Find the particular solution of the given differential equation.

$$\frac{dy}{dx} = xe^{y-x^2} \; ; \quad y=0 \text{ when } x=1$$

Rewrite: $dy = xe^{y}e^{-x^2}dx$

$$e^{-y}\,dy = xe^{-x^2}dx$$

$$\underbrace{\int e^{-y}dy}_{} = \underbrace{\int xe^{-x^2}dx}_{}$$

$u=-y$ $\quad$ $v=-x^2$

$du=-dy$ $\quad$ $dv=-2x\,dx$

$-du=dy$ $\quad$ $\frac{dv}{-2}=x\,dx$

$$\int -e^{u}du = \int -\frac{1}{2}e^{v}dv$$

$$-e^{u} = -\frac{1}{2}e^{v} + c$$

$$-e^{-y} = \frac{-1}{2}e^{-x^2} + c$$

$$e^{-y} = \frac{1}{2}e^{-x^2} - C$$

$$e^{-y} = \frac{1}{2}e^{-x^2} + C$$

$$\ln e^{-y} = \ln\left(\frac{1}{2}e^{-x^2} + C\right)$$

$$-y = \ln\left(\frac{1}{2}e^{-x^2} + C\right)$$

$$y = -\ln\left(\frac{1}{2}e^{-x^2} + C\right)$$