Lesson 7: Separation of Variables

Recall from Last Time,

If a differential equation can be written in the form

$$g(y)\,dy = h(x)\,dx \quad \text{from} \quad \frac{dy}{dx} = \frac{h(x)}{g(y)}$$

$$\int g(y)\,dy = \int h(x)\,dx$$

$$G(y) + C_1 = H(x) + C_2$$

$$G(y) = H(x) + C_2 - C_1$$

$$G(y) = H(x) + C$$

<u>Example 1</u>: Solve the differential equation: $\frac{dy}{dt} = 2t(25-y)$

Rewrite: $\frac{dy}{25-y} = 2t\,dt$

$$\underbrace{\int \frac{dy}{25-y}} = \int 2t\,dt$$

$u = 25-y$

$du = -dy$

$-du = dy$

$$\int \frac{-du}{u} = \int 2t\,dt$$

$$-\ln|u| = \frac{2t^2}{2} + C$$

$$-\ln|25-y| = t^2 + C$$

$$\ln|25-y|^{-1} = t^2 + C$$

$$\ln\left|\frac{1}{25-y}\right| = t^2 + C$$

$$\exp\left[\ln\left|\frac{1}{25-y}\right|\right] = \exp[t^2 + C]$$

$$\pm\frac{1}{25-y} = e^{t^2}e^{C}$$

$$\frac{1}{25-y} = \pm e^{C}e^{t^2}$$

$$\frac{1}{25-y} = Ce^{t^2}$$

$$25-y = \frac{1}{Ce^{t^2}}$$

$$25-y = \frac{1}{C}e^{-t^2} = Ce^{-t^2}$$

$$25 - Ce^{-t^2} = y$$

Example 2: Find the general solution of the differential equation: $y' = xy\sin(x^2)$.

Recall $y' = dy/dx$. So

$$\frac{dy}{y} = x\sin(x^2)\,dx$$

$$\int \frac{dy}{y} = \underbrace{\int x\sin(x^2)\,dx}$$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int \frac{dy}{y} = \int \sin(u)\,\frac{du}{2}$$

$$\ln|y| = -\frac{\cos(u)}{2} + C$$

$$\ln|y| = \frac{-1}{2}\cos(x^2) + C$$

$$\exp\left[\ln|y|\right] = \exp\left[\frac{-1}{2}\cos(x^2) + C\right]$$

$$\pm y = e^C \exp\left[\frac{-1}{2}\cos(x^2)\right]$$

$$y = \pm e^C \exp\left[\frac{-1}{2}\cos(x^2)\right]$$

$$y = C\exp\left[-\frac{1}{2}\cos(x^2)\right]$$

Example 3: Find the particular solution of the differential equation: $\frac{dx}{dt} = xt\sqrt{t+1}$; $x=1$ when $t=0$

Rewrite: $\frac{dx}{x} = t(t+1)^{1/2}dt$

$$\int \frac{dx}{x} = \underbrace{\int t(t+1)^{1/2}\,dt}_{u=t+1 \iff t=u-1}$$

$du = dt$

$$\int \frac{dx}{x} = \int (u-1)u^{1/2}\,du$$

$$\int \frac{dx}{x} = \int u^{3/2} - u^{1/2}\,du$$

$$\ln|x| = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$\ln|x| = \frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2} + C$$

$$\exp\left[\ln|x|\right] = \exp\left[\frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2} + C\right]$$

$$\pm x = e^{C} \cdot \exp\left[\frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2}\right]$$

$$x = \pm e^{C}\exp\left[\frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2}\right]$$

$$x = C\exp\left[\frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2}\right]$$

Now find C w/ $x=1$ when $t=0$.

$$1 = C\exp\left[\frac{2}{5} - \frac{2}{3}\right]$$

$$1 = C\exp\left[-\frac{4}{15}\right]$$

$$C = \exp\left[\frac{4}{15}\right]$$

So $x = \exp\left[\frac{4}{15}\right] \exp\left[\frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2}\right]$

$= \exp\left[\frac{4}{15} + \frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2}\right]$