Lesson 8: Separation of Variables

From the Problem Set (posted online),

Example 1: Find the general solution of the differential equation:

$$5x^3y + y' = 2x^3$$

Rewrite: $y' = 2x^3 - 5x^3y$

$$\frac{dy}{dx} = x^3(2-5y)$$

$$\frac{dy}{2-5y} = x^3\,dx$$

$$\underbrace{\int \frac{dy}{2-5y}}_{u=2-5y} = \int x^3\,dx$$

$du = -5dy$

$\frac{du}{-5} = dy$

$$\int \frac{1}{u}\cdot\frac{du}{-5} = \int x^3\,dx$$

$$-\frac{1}{5}\ln|u| = \frac{x^4}{4} + C$$

$$-\frac{1}{5}\ln|2-5y| = \frac{x^4}{4} + C$$

$$\ln|2-5y|^{-1/5} = \frac{x^4}{4} + C$$

$$\exp\left[\ln\left|\frac{1}{(2-5y)^{1/5}}\right|\right] = \exp\left[\frac{x^4}{4} + C\right]$$

$$\pm\frac{1}{(2-5y)^{1/5}} = e^C\exp\left[\frac{x^4}{4}\right]$$

$$\frac{1}{(2-5y)^{1/5}} = \pm e^C\exp\left[\frac{x^4}{4}\right]$$

$$\frac{1}{(2-5y)^{1/5}} = Ce^{x^4/4}$$

$$(2-5y)^{1/5} = \frac{1}{Ce^{x^4/4}} = \frac{1}{C}\exp[-x^4/4]$$

$$(2-5y)^{1/5} = C\exp[-x^4/4]$$

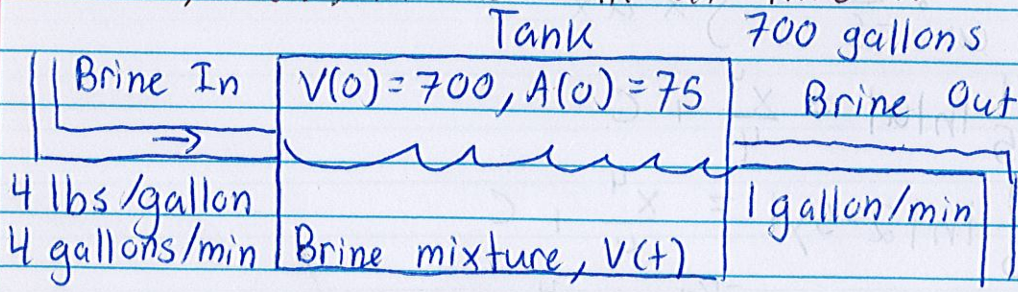
$$2-5y = C^5 \exp\left[\frac{-5}{4}x^4\right]$$

$$2-5y = C\exp\left[\frac{-5}{4}x^4\right]$$

$$2 - C\exp\left[\frac{-5}{4}x^4\right] = 5y$$

$$y = \frac{1}{5}\left(2 - C\exp\left[\frac{-5}{4}x^4\right]\right)$$

<u>Example 2:</u> A 800 gallon tank initially contains 700 gallons of brine containing 75 pounds of dissolved salt. Brine containing 4 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 1 gallon per minute. Set up a differential equation for the amount of salt, $A(t)$, in the tank at time t.



<u>Define:</u> $V(t)$ = amount of brine mixture in tank at time t (in gallons)

$A(t)$ = amount of salt in the tank at time t (in pounds)

t = time in minutes

$$\frac{dA}{dt} = \begin{pmatrix}\text{rate of change of}\\ \text{salt in tank in lbs/min}\end{pmatrix} = \begin{pmatrix}\text{rate in}\\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{rate out}\\ \text{of salt}\end{pmatrix}$$

Rate in: $\left(4 \, \frac{\text{lbs}}{\text{gallons}}\right)\left(4 \, \frac{\text{gallons}}{\text{min}}\right) = 16 \, \frac{\text{lbs}}{\text{min}}$

Rate out: "Well-stirred" means each gallon in the tank has as much salt in it as any other gallon. i.e. Salt is uniformly mixed in the brine mixture

$$= \left(\begin{matrix}\text{salt out per gallon} \\ \text{of brine mixture}\end{matrix}\right)\left(\begin{matrix}\text{mixture out} \\ \text{per minute}\end{matrix}\right)$$

$$= \left(\frac{A(t)}{V(t)} \, \frac{\text{lbs}}{\text{gallons}}\right)\left(1 \, \frac{\text{gallons}}{\text{mins}}\right) = \frac{A(t)}{V(t)} \, \frac{\text{lbs}}{\text{min}}$$

$\frac{dA}{dt} = 16 - \frac{A(t)}{V(t)}$. Now find $V(t)$.

So $\frac{dV}{dt} = \left(\begin{matrix}\text{rate of change of} \\ \text{brine mix in gallons/min}\end{matrix}\right) = \left(\begin{matrix}\text{rate in} \\ \text{of brine}\end{matrix}\right) - \left(\begin{matrix}\text{rate out} \\ \text{of brine}\end{matrix}\right)$

$$= 4 \, \frac{\text{gallons}}{\text{min}} - 1 \, \frac{\text{gallons}}{\text{min}} = 3 \, \frac{\text{gallons}}{\text{min}}$$

Hence $\begin{cases} V'(t) = 3 \\ V(0) = 700 \end{cases}$

$$V(t) = \int V'(t)\,dt = \int 3\,dt = 3t + C$$

when $V(0) = 700$,

$$700 = V(0) = C \Rightarrow V(t) = 3t + 700$$

Hence $\frac{dA}{dt} = 16 - \frac{A}{3t + 700}$

Example 3: In a particular chemical reaction, a substance is converted into a second substance at a rate proportional to the square of the amount of the first substance present at any time, t. Initially, 50 grams of the first substance was present, and 1 hour later only 14 grams of the first substance remained. What is the amount of the first substance remaining after 7 hours?

Set-Up: $\frac{da}{dt} = a^2k$; $a(0)=50$; $a(1)=14$

Solve: $\frac{da}{a^2} = k\,dt$

$\int a^{-2}\,da = \int k\,dt$

$-a^{-1} = kt + C$

$-\frac{1}{a} = kt + C$

$\frac{1}{a} = -kt + C$

$a = \frac{1}{-kt + C}$

When $a(0)=50$,

$50 = a(0) = \frac{1}{C}$

$\frac{1}{50} = C \quad \Rightarrow a = \frac{1}{1/50 - kt} = \frac{50}{1-50kt}$

When $a(1) = 14$,

$14 = a(1) = \frac{50}{1-50k}$

$14(1-50k) = 50$

$1 - 50k = \frac{50}{14} = \frac{25}{7}$

$-50k = \frac{25}{7} - 1 = \frac{18}{7}$

$k = \frac{-1}{50} \cdot \frac{18}{7} = -\frac{18}{350}$

$$\text{So} \quad a = \frac{50}{1 - 50\left(\frac{-18}{350}\right)t} = \frac{50}{1 + \frac{18}{7}t} = \frac{350}{7 + 18t}$$

$$\text{Hence } a(7) = \frac{350}{7 + 18(7)} \approx 2.6316 \text{ grams}$$