Lesson 9: First-Order Linear Differential Equations

Let's define what a First-Order Linear Differential Equation is.

First-order means that only the first derivative appears (so no y'', y''', etc.)

Linear means that y' and y are not multiplied together in any combination.

example: $y' + ty = t^2 + 6$

NOT example: $yy' + y = 1$

Differential Equation is an equation that relates one or more functions and their derivatives.

A first order linear equation can be written in the standard form:

$$y' + P(t)\,y = Q(t) \qquad (*)$$

Why do we want it in this form?

Let $u(t) = \exp[\int P(t)\,dt]$. Multiply both sides of $(*)$ by $u(t)$, we get

$$y'\exp[\int P(t)\,dt] + P(t)\,y\exp[\int P(t)\,dt] = Q(t)\exp[\int P(t)\,dt]$$

By Warm-Up 3, the LHS is

$$\frac{d}{dt}\left(y\exp[\int P(t)\,dt]\right) = Q(t)\exp[\int P(t)\,dt]$$

Remember $u(t) = \exp[\int P(t)\,dt]$. So

$$\frac{d}{dt}(y\,u(t)) = Q(t)\,u(t)$$

$$\int \frac{d}{dt}(y\,u(t))\,dt = \int Q(t)\,u(t)\,dt$$

$$y\,u(t) = \int Q(t)u(t)\,dt$$

<u>Definition:</u> The term $u(t) = \exp[\int P(t)dt]$ is called an integrating factor.

<u>To summarize:</u>

Given an equation of the form

$$y' + P(t)\,y = Q(t)$$

a solution is given by

$$y\,u(t) = \int Q(t)\,u(t)\,dt$$

where $u(t) = \exp[\int P(t)dt]$.

<u>How to solve First-Order Linear Equations</u>

In general, say you are given

$$a(t)\,y' + b(t)\,y = c(t)$$

Then

① Divide everything by $a(t)$, assuming $a(t) \neq 0$. So

$$y' + \frac{b(t)}{a(t)}\,y = \frac{c(t)}{a(t)}$$

i.e. We are getting the equation into Standard Form.

② Determine $P(t)$ and $Q(t)$

$$P(t) = \frac{b(t)}{a(t)} \qquad Q(t) = \frac{c(t)}{a(t)}$$

③ Find integrating factor. i.e. $u(t) = \exp[\int P(t)\,dt]$

④ Plug $u(t)$ and $Q(t)$ in

$$y \cdot u(t) = \int Q(t)\,u(t)\,dt$$

⑤ Integrate the RHS of ④

⑥ Divide both sides of eqn from ⑤ by $u(t)$.

Example 1: Find the general solution of

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = 3x - 5$$

Since the ode in standard form, ① ✓

② $P(x) = \frac{2}{x}$ $\quad Q(x) = 3x - 5$

③ $u(x) = \exp[\int P(x)dx] = \exp\left[\int \frac{2}{x}dx\right] = \exp\left[2\ln(x)\right]$

$= \exp[\ln x^2] = x^2$

④ $y\,u(x) = \int Q(x)u(x)\,dx$

$y\,x^2 = \int (3x-5)\,x^2\,dx$

⑤ $y\,x^2 = \int (3x^3 - 5x^2)\,dx$

$yx^2 = \frac{3x^4}{4} - \frac{5x^3}{3} + C$

⑥ $y = \frac{3}{4}\frac{x^4}{x^2} - \frac{5}{3}\frac{x^3}{x^2} + \frac{C}{x^2}$

$= \frac{3}{4}x^2 - \frac{5}{3}x + \frac{C}{x^2}$

Example 2: Solve the IVP $\begin{cases} t^2y' + ty = 7 \\ y(1) = 7 \end{cases}$

① $t^2y' + ty = 7$

$\frac{t^2y'}{t^2} + \frac{ty}{t^2} = \frac{7}{t^2}$

$y' + \frac{1}{t}y = \frac{7}{t^2}$

② $P(t) = \frac{1}{t}$ $\quad Q(t) = \frac{7}{t^2}$

③ $u(t) = \exp[\int P(t)\,dt] = \exp\left[\int \frac{1}{t}\,dt\right] = \exp[\ln t] = t$

④ $y \cdot u(t) = \int Q(t)u(t)\,dt$

$y \cdot t = \int \frac{7}{t^2} \cdot t\,dt$

⑤ $yt = \int \frac{7}{t}\,dt$

$yt = 7\ln|t| + C$

⑥ $y = \frac{7\ln|t|}{t} + \frac{C}{t}$

But since this is an IVP, we need to do more work.

When $y(1) = 7$,

$7 = \frac{7\ln|1|}{1} + \frac{C}{1}$

$7 = 0 + C$

$C = 7 \implies y = \frac{7\ln|t|}{t} + \frac{7}{t}$

Example 3: Find the general solution of $y' - y = 19$

Since the ode in standard form, ① ✓

② $P(t) = -1 \quad Q(t) = 19$

③ $u(t) = \exp[\int P(t)dt] = \exp[\int -dt] = \exp[-t] = e^{-t}$

④ $y \cdot u(t) = \int Q(t)u(t)\,dt$

$y e^{-t} = \int 19 e^{-t}\,dt$

⑤ $y e^{-t} = -19e^{-t} + C$

⑥ $y = \dfrac{-19e^{-t}}{e^{-t}} + \dfrac{C}{e^{-t}}$

$y = -19 + Ce^{t}$

Note: You can do this problem via Separation of Variable. Left as an exercise for the reader.

Example 4: Find the general solution of

$$y' + \left(\frac{1}{x}\right)y = \sin(x^2) \quad \text{on } (\sqrt{\pi}, 0)$$

Since the ode in standard form, ① ✓

② $P(x) = \dfrac{1}{x}$ $\qquad Q(x) = \sin(x^2)$

③ $u(x) = \exp[\int P(x)\,dx] = \exp\left[\int \frac{1}{x}\,dx\right] = \exp[\ln x] = x$

④ $y \cdot u(x) = \int Q(x)u(x)\,dx$

$yx = \int x\sin(x^2)\,dx$

⑤ Let's do a u-substitution on RHS.

$u = x^2 \qquad du = 2x\,dx$

$$yx = \int \sin u \, \frac{du}{2}$$

$$yx = -\frac{\cos u}{2} + C$$

$$yx = -\frac{\cos(x^2)}{2} + C$$

(6) $$y = -\frac{\cos(x^2)}{2x} + \frac{C}{x}$$