Lesson 10: Higher Order Derivatives

The derivative of a function, $f(x)$, is also called the <u>first derivative</u>.

$$y', \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)]$$

If we take the derivative of the first derivative of $y=f(x)$, then we get the <u>second derivative</u>.

$$y'', \quad f''(x), \quad \frac{d^2y}{dx^2}, \quad \frac{d^2}{dx^2}[f(x)]$$

Take the derivative n times, we get the <u>n-th derivative</u>.

$$y^{(n)}, \quad f^{(n)}(x), \quad \frac{d^n x}{dx^n}, \quad \frac{d^n}{dx^n}[f(x)]$$

<u>Example 1</u>: Find the first three derivatives of the following functions:

ⓐ $y = 3x^2 + 8x^{1/2} + e^x$

$$y' = 3(2)x + 8(\tfrac{1}{2})x^{-1/2} + e^x$$
$$= 6x + 4x^{-1/2} + e^x$$

$$y'' = 6 + 4(-\tfrac{1}{2})x^{-3/2} + e^x$$
$$= 6 - 2x^{-3/2} + e^x$$

$$y''' = 0 - 2(-\tfrac{3}{2})x^{-5/2} + e^x$$
$$= 3x^{-5/2} + e^x$$

ⓑ $y = xe^{2x}$

Let $u(x) = x \qquad v(x) = e^{2x}$

$u'(x) = 1 \qquad v'(x) = 2e^{2x}$

By Product Rule,

$$y' = u'(x)v(x) + u(x)v'(x)$$
$$= 1 \cdot e^{2x} + x \cdot 2e^{2x}$$
$$= (1+2x)e^{2x}$$

Let $u(x) = 1+2x \qquad v(x) = e^{2x}$

$u'(x) = 2 \qquad v'(x) = 2e^{2x}$

By Product Rule,

$$y'' = u'(x)v(x) + u(x)v'(x)$$
$$= 2 \cdot e^{2x} + (1+2x) \cdot 2e^{2x}$$
$$= 2 \cdot e^{2x} + (2+4x) \cdot e^{2x}$$
$$= (2+2+4x)e^{2x}$$
$$= (4+4x)e^{2x}$$

Let $u(x) = 4+4x$ $\quad v(x) = e^{2x}$

$u'(x) = 4$ $\quad v'(x) = 2e^{2x}$

By Product Rule,

$$y''' = u'(x)v(x) + u(x)v'(x)$$
$$= 4 \cdot e^{2x} + (4+4x) \cdot 2e^{2x}$$
$$= 4 \cdot e^{2x} + (8+8x) \cdot e^{2x}$$
$$= (4+8+8x)e^{2x}$$
$$= (12+8x)e^{2x}$$

Example 2: Find the second derivative of $y = 10x^2 \ln(3x)$

Let $u(x) = 10x^2$ $\quad v(x) = \ln(3x)$

$u'(x) = 20x$ $\quad v'(x) = \frac{1}{3x} \cdot 3 = \frac{1}{x}$

By Product Rule,

$$y' = u'(x)v(x) + u(x)v'(x)$$
$$= 20x \ln(3x) + 10x^2 \cdot \frac{1}{x}$$
$$= \underbrace{20x \ln(3x)}_{(*)} + 10x$$

To find the derivative of $(*)$, we need to apply product rule again.

Let $u(x) = 20x$ $\quad v(x) = \ln(3x)$

$u'(x) = 20$ $\quad v'(x) = 1/x$

By Product Rule,

$$(*)' = u'(x)v(x) + u(x)v'(x)$$
$$= 20 \ln(3x) + 20x \cdot \frac{1}{x}$$
$$= 20 \ln(3x) + 20$$

So $y'' = 20\ln(3x) + 20 + 10$
$= 20\ln(3x) + 30$

Position/Velocity/Acceleration Functions

Recall that $v(t) = s'(t)$

Acceleration function, $a(t)$, tells us how fast the velocity change. Hence

$$a(t) = v'(t) = (s'(t))' = s''(t)$$

Example 3: A particle is traveling on a straight line with a position function of

$$s(t) = t^3 + 12t^2$$

ⓐ What is the particle's acceleration? i.e. $s''(t)$

$v(t) = s'(t) = 3t^2 + 12(2)t$
$= 3t^2 + 24t$

$a(t) = s''(t) = 6t + 24$

ⓑ What's the acceleration when the velocity of the particle is 384?
i.e. Solve $v(t) = 384$ for t, and then plug that t into $a(t)$.

$v(t) = 384$
$3t^2 + 24t = 384$
$3t^2 + 24t - 384 = 0$
$3(t^2 + 8t - 128) = 0$
$3(t+16)(t-8) = 0$

$t + 16 = 0$ | $t - 8 = 0$
$\cancel{t = -16}$ | $t = 8$

We can ignore $t = -16$ b/c time can't be negative.

So $a(8) = 6(8) + 24$
$= 32 + 24$
$= 56$

Implicit Differentiation Pt 1

Explicit Form: $y = f(x)$

Implicit Form: When a function is NOT written in explicit form

ex. ① $y - 2x = 1$ ② $x^2 + y^2 = 2$ ③ $y^2 + y - 1 = x$

To differentiate functions of this kind, we use a technique called implicit differentiation.

We namely use this technique when solving for y is particularly messy.

Example 1: Use implicit differentiation to find dy/dx of the following functions:

ⓐ $7y^2 = 15 + 6x^2$

Take the derivative with respect to x on both sides.

$$\frac{d}{dx}(7y^2) = \frac{d}{dx}(15 + 6x^2)$$

$$\frac{d}{dx}(7y^2) = \frac{d}{dx}(15) + \frac{d}{dx}(6x^2)$$

$$14y\frac{dy}{dx} = 0 + 12x\frac{dx}{dx}$$

$$14y\frac{dy}{dx} = 12x$$

Now solve for dy/dx.

$$\frac{dy}{dx} = \frac{12x}{14y}$$

$$\frac{dy}{dx} = \frac{6x}{7y}$$

ⓑ $6x^2 + 19xy + 8y^2 = 14$

Take the derivative with respect to x on both sides.

$$\frac{d}{dx}(6x^2 + 19xy + 8y^2) = \frac{d}{dx}(14)$$

$$\frac{d}{dx}(6x^2) + 19\frac{d}{dx}(xy) + \frac{d}{dx}(8y^2) = \frac{d}{dx}(14)$$

$$12x\frac{dx}{dx} + 19\frac{d}{dx}(xy) + 16y\frac{dy}{dx} = 0$$

$$12x + 19\frac{d}{dx}(xy) + 16y\frac{dy}{dx} = 0$$

Is this a function of x or y? Both
Is it a product or quotient? Product ⇒ Use Product Rule

$$12x + 19\left(\frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y)\right) + 16y\frac{dy}{dx} = 0$$

$$12x + 19\left(1\cdot\frac{dx}{dx}\cdot y + x\cdot 1\cdot\frac{dy}{dx}\right) + 16y\frac{dy}{dx} = 0$$

$$12x + 19\left(y + x\frac{dy}{dx}\right) + 16y\frac{dy}{dx} = 0$$

$$12x + 19y + 19x\frac{dy}{dx} + 16y\frac{dy}{dx} = 0$$

Now solve for dy/dx.

$$19x\frac{dy}{dx} + 16y\frac{dy}{dx} = -12x - 19y$$

$$(19x + 16y)\frac{dy}{dx} = -12x - 19y$$

$$\frac{dy}{dx} = \frac{-12x - 19y}{19x + 16y}$$

Extra Credit 2: Let y be a function of x. Use implicit differentiation on the following expressions.

ⓐ $$\frac{d}{dx}(xy) = \frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y)$$

$$= 1\cdot\frac{dx}{dx}\cdot y + x\cdot 1\cdot\frac{dy}{dx}$$

$$= y + x\frac{dy}{dx}$$

Basically it's product rule.

ⓑ $$\frac{d}{dx}\left(\frac{x}{y}\right)$$

Hint use quotient rule.