Lesson 11: Implicit Differentiation Pt 2

Recall from Last Time the method of implicit differentiation.

↳ We use it when solving for y is messy and difficult.

↳ When finding dy/dx via this method, we do the following

- Take $\frac{d}{dx}$ to both sides of the equation, and
- Then solve for dy/dx.

Example 1: ⓐ Find the slope of the tangent line of $2x^4 = 4y^2 + 4x^2$ at $(2, -2)$

i.e. Find $\frac{dy}{dx}$ and plug in $\begin{cases} x = 2 \\ y = -2 \end{cases}$

Take d/dx to both sides and solve for dy/dx.

$$\frac{d}{dx}(2x^4) = \frac{d}{dx}(4y^2 + 4x^2)$$

$$\frac{d}{dx}(2x^4) = \frac{d}{dx}(4y^2) + \frac{d}{dx}(4x^2)$$

$$8x^3 \cancel{\frac{dx}{dx}} = 8y\frac{dy}{dx} + 8x\cancel{\frac{dx}{dx}}$$

$$8x^3 = 8y\frac{dy}{dx} + 8x$$

$$8x^3 - 8x = 8y\frac{dy}{dx}$$

$$\frac{8x^3 - 8x}{8y} = \frac{dy}{dx}$$

$$\frac{\cancel{8}(x^3 - x)}{\cancel{8}y} = \frac{dy}{dx}$$

$$\frac{x^3 - x}{y} = \frac{dy}{dx}$$

Plug $x = 2$, $y = -2$ into dy/dx.

$$\frac{dy}{dx} = \frac{2^3 - 2}{2} = \frac{8-2}{2} = \frac{6}{2} = 3$$

ⓑ Using ⓐ, find the equation of the tangent line at $(2,-2)$.

Note we have our "slope" from ⓐ, and we are given a point, so all we need to do is put it in point-slope formula,

$$y-(-2) = 3(x-2)$$
$$y+2 = 3x-6$$
$$-2 \qquad\quad -2$$
$$y = 3x-8$$

Example 2: Use implicit differentiation to find dy/dx of the following functions:

ⓐ $4\sin x \cos y = 3$

$$\frac{d}{dx}(4\sin x \cos y) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}(4\sin x)\cdot\cos y + 4\sin x\cdot\frac{d}{dx}(\cos y) = \frac{d}{dx}(3)$$

$$4\cos x\,\frac{dx}{dx}\cos y + 4\sin x\cdot(-\sin y)\frac{dy}{dx} = 0$$

$$4\cos x\cos y - 4\sin x\sin y\,\frac{dy}{dx} = 0$$

$$4\cos x\cos y = 4\sin x\sin y\,\frac{dy}{dx}$$

$$\frac{\cancel{4}\cos x\cos y}{\cancel{4}\sin x\sin y} = \frac{dy}{dx}$$

$$\frac{\cos x}{\sin x}\cdot\frac{\cos y}{\sin y} = \frac{dy}{dx}$$

$$\cot x\ \cot y = \frac{dy}{dx}$$

ⓑ $6\tan(2x+3y) = 11xy$

$$\frac{d}{dx}\left(6\tan(2x+3y)\right) = \frac{d}{dx}(11xy)$$

$$6\sec(2x+3y) \cdot \frac{d}{dx}(2x+3y) = \frac{d}{dx}(11x) \cdot y + 11x\frac{d}{dx}(y)$$

$$6\sec(2x+3y)\left[2\frac{\cancel{dx}}{\cancel{dx}} + 3\frac{dy}{dx}\right] = 11\frac{\cancel{dx}}{\cancel{dx}} \cdot y + 11x\frac{dy}{dx}$$

$$6\sec(2x+3y)\left[2 + 3\frac{dy}{dx}\right] = 11y + 11x\frac{dy}{dx}$$

$$12\sec(2x+3y) + 18\sec(2x+3y)\frac{dy}{dx} = 11y + 11x\frac{dy}{dx}$$

$$12\sec(2x+3y) - 11y = 11x\frac{dy}{dx} - 18\sec(2x+3y)\frac{dy}{dx}$$

$$12\sec(2x+3y) - 11y = \left(11x - 18\sec(2x+3y)\right)\frac{dy}{dx}$$

$$\frac{12\sec(2x+3y) - 11y}{11x - 18\sec(2x+3y)} = \frac{dy}{dx}$$

ⓒ $2\sin\left(\frac{x}{y}\right) = 9x$

$$\frac{d}{dx}\left(2\sin\left(\frac{x}{y}\right)\right) = \frac{d}{dx}(9x)$$

$$2\cos\left(\frac{x}{y}\right) \cdot \frac{d}{dx}\left(\frac{x}{y}\right) = 9\frac{dx}{dx}$$

$$2\cos\left(\frac{x}{y}\right) \cdot \left[\frac{\frac{d}{dx}(x) \cdot y - x \cdot \frac{d}{dx}(y)}{y^2}\right] = 9$$

$$\frac{2\cos(x/y)}{y^2}\left[\frac{dx}{dx} \cdot y - x\frac{dy}{dx}\right] = 9$$

$$\frac{2\cos(x/y)}{y^2}\left[y - x\frac{dy}{dx}\right] = 9$$

$$y - x\frac{dy}{dx} = \frac{9y^2}{2\cos(x/y)}$$

$$-x\frac{dy}{dx} = \frac{9y^2}{2\cos(x/y)} - y$$

$$\frac{dy}{dx} = \frac{-1}{x}\left(\frac{9y^2}{2\cos(x/y)} - y\right)$$

$$\frac{dy}{dx} = \frac{-9y^2}{2x\cos(x/y)} + \frac{y}{x}$$

Formal Proof of $\frac{d}{dx}[\ln x] = \frac{1}{x}$

Let $y = \ln x$. Note that $y = \ln x \iff e^y = x$.

Differentiate.

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y\frac{dy}{dx} = 1\frac{dx}{dx}$$

$$e^y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Recall that, So

$$\frac{dy}{dx} = \frac{1}{x}$$

Hence

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

MA 16010 LESSON 11+12: RELATED RATES HANDOUT

Related Rates are word problems that use implicit differentiation.

We will be taking the derivative of equations with respect to time, t.

Recipe for Solving a Related Rates Problem

Step 1: Draw a good picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry, similar triangles, the Pythagorean Theorem, or a formula from trigonometry. **Use your picture!**

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**. Do **NOT** substitute before this step!

Some Useful Formulas

Right Triangle *Pythagorean Theorem:* $a^2+b^2=c^2$	Triangle $A=\frac{1}{2}bh$ $P=a+b+c$	Trapezoid $A=\frac{1}{2}(a+b)h$	Rectangular Box $V=lwh$ $S=2(hl+lw+hw)$	Cone $V=\frac{1}{3}\pi r^2 h$ $SA=\pi rl+\pi r^2$
Rectangle $A=lw$ $P=2l+2w$	Circle $A=\pi r^2$ $C=2\pi r$	Sphere $A=\frac{4}{3}\pi r^3$ $S=4\pi r^2$	Right Circular Cylinder $V=\pi r^2 h$ $SA=2\pi rh$	

Example 1: If x and y are both functions of t and $x + y^3 = 2$.

a) Find $\frac{dy}{dt}$ when $\frac{dx}{dt} = -2$ and $y = 1$

$$\frac{d}{dt}(x+y^3) = \frac{d}{dt}(2)$$

$$\frac{d}{dt}(x) + \frac{d}{dt}(y^3) = \frac{d}{dt}(2)$$

$$1 \cdot \frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

Now plug $\frac{dx}{dt} = -2$ and $y = 1$

$$-2 + 3(1)^2\frac{dy}{dt} = 0$$

$$3\frac{dy}{dt} = 2$$

$$\frac{dy}{dt} = \frac{2}{3}$$

b) Find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 3$ and $x = 1$

From a, we found

$$\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

Plug $\frac{dy}{dt} = 3$ and $x = 1$

$$\frac{dx}{dt} + 3y^2 \cdot 3 = 0$$

$$\frac{dx}{dt} = -9y^2$$

But I want a #. So we can find y by plugging $x=1$ into the original eqn.

$$1 + y^3 = 2$$

$$y^3 = 1$$

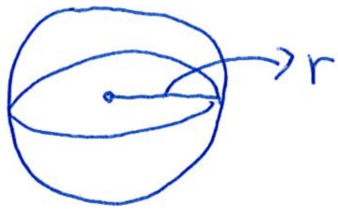
$$y = 1$$

Hence

$$\frac{dx}{dt} = -9 \cdot 1^2 = -9$$

Example 2: **A spherical balloon is being deflated at a constant rate of 20 cubic cm per second. How fast is the radius of the balloon changing at the instant when the balloon's radius is 12 cm?**

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -20 \frac{cm^3}{s}$ **WANT:** $\left.\frac{dr}{dt}\right|_{r=12}$

Step 3: Find an equation relating the relevant variables.

$$V = \frac{4}{3}\pi r^3$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

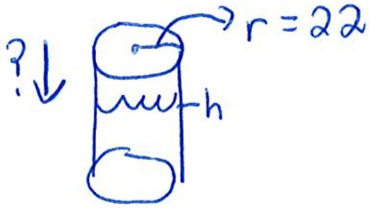
$$-20 = 4\pi(12)^2 \frac{dr}{dt}$$

$$\frac{-20}{4\pi(12)^2} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-5}{144\pi}$$

Example 3: A cylindrical tank standing upright (with one circular base on the ground) has a radius of 22 cm for the base. How fast does the water level in the tank drop when the water is being drained at 28 cm^3/sec? Note: The formula right circular cylinder is $V = \pi r^2 h$.

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $r = 22$ and $\frac{dV}{dt} = -28 \frac{cm^3}{s}$ **WANT:** $\frac{dh}{dt}$

Step 3: Find an equation relating the relevant variables.

$$V = \pi r^2 h \iff V = \pi (22)^2 h$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\pi \cdot (22)^2 h\right)$$

$$\frac{dV}{dt} = \pi \cdot (22)^2 \cdot \frac{dh}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$-28 = \pi \cdot (22)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-28}{\pi \cdot (22)^2}$$

$$\frac{dh}{dt} = \frac{-7}{121\pi}$$