MA 16010 LESSON 11+12: RELATED RATES HANDOUT

Related Rates are word problems that use implicit differentiation.

We will be taking the derivative of equations with respect to time, t.

Recipe for Solving a Related Rates Problem

Step 1: Draw a good picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry, similar triangles, the Pythagorean Theorem, or a formula from trigonometry. **Use your picture!**

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**. Do **NOT** substitute before this step!

Some Useful Formulas

Right Triangle *Pythagorean Theorem:* $a^2 + b^2 = c^2$	Triangle $A = \frac{1}{2}bh$ $P = a + b + c$	Trapezoid $A = \frac{1}{2}(a+b)h$	Rectangular Box $V = lwh$ $S = 2(hl + lw + hw)$	Cone $V = \frac{1}{3}\pi r^2 h$ $SA = \pi r l + \pi r^2$
Rectangle $A = lw$ $P = 2l + 2w$	Circle $A = \pi r^2$ $C = 2\pi r$	Sphere $A = \frac{4}{3}\pi r^3$ $S = 4\pi r^2$	Right Circular Cylinder $V = \pi r^2 h$ $SA = 2\pi r h$	

Example 1: If x and y are both functions of t and $x + y^3 = 2$.

a) Find $\frac{dy}{dt}$ when $\frac{dx}{dt} = -2$ and $y = 1$

$$\frac{d}{dt}(x+y^3) = \frac{d}{dt}(2)$$

$$\frac{d}{dt}(x) + \frac{d}{dt}(y^3) = \frac{d}{dt}(2)$$

$$1 \cdot \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0$$

Now plug $\frac{dx}{dt} = -2$ and $y = 1$

$$-2 + 3(1)^2 \frac{dy}{dt} = 0$$

$$3\frac{dy}{dt} = 2$$

$$\frac{dy}{dt} = \frac{2}{3}$$

b) Find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 3$ and $x = 1$

From a, we found

$$\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

Plug $\frac{dy}{dt} = 3$ and $x = 1$

$$\frac{dx}{dt} + 3y^2 \cdot 3 = 0$$

$$\frac{dx}{dt} = -9y^2$$

But I want a #. So we can find y by plugging $x=1$ into the original eqn.

$$1 + y^3 = 2$$

$$y^3 = 1$$

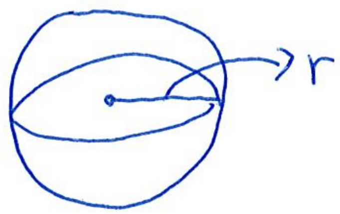
$$y = 1$$

Hence

$$\frac{dx}{dt} = -9 \cdot 1^2 = -9$$

Example 2: **A spherical balloon is being deflated at a constant rate of 20 cubic cm per second. How fast is the radius of the balloon changing at the instant when the balloon's radius is 12 cm?**

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -20 \frac{cm^3}{s}$ **WANT:** $\left.\frac{dr}{dt}\right|_{r=12}$

Step 3: Find an equation relating the relevant variables.

$$V = \frac{4}{3}\pi r^3$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

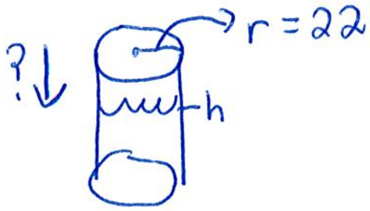
$$-20 = 4\pi(12)^2 \frac{dr}{dt}$$

$$\frac{-20}{4\pi(12)^2} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-5}{144\pi}$$

Example 3: A cylindrical tank standing upright (with one circular base on the ground) has a radius of 22 cm for the base. How fast does the water level in the tank drop when the water is being drained at 28 cm^3/sec? Note: The formula right circular cylinder is $V = \pi r^2 h$.

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $r = 22$ and $\frac{dV}{dt} = -28 \frac{cm^3}{s}$ **WANT:** $\frac{dh}{dt}$

Step 3: Find an equation relating the relevant variables.

$$V = \pi r^2 h \quad \Longleftrightarrow \quad V = \pi (22)^2 h$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\pi \cdot (22)^2 h\right)$$

$$\frac{dV}{dt} = \pi \cdot (22)^2 \cdot \frac{dh}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$-28 = \pi \cdot (22)^2 \frac{dh}{dt}$$

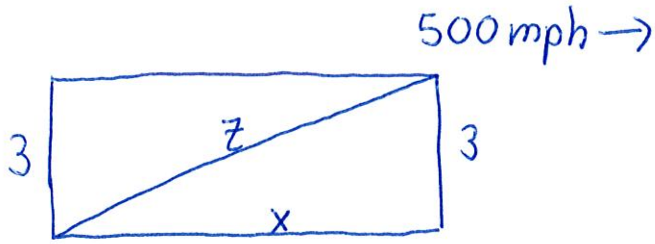
$$\frac{dh}{dt} = \frac{-28}{\pi \cdot (22)^2}$$

$$\frac{dh}{dt} = \frac{-7}{121\pi}$$

Example 4: **A plane is flying directly away from you at 500 mph at an altitude of 3 miles.**

a) How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500\text{mph}$ **WANT:** $\left.\frac{dz}{dt}\right|_{x=4}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + 3^2 = z^2 \iff x^2 + 9 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2+9) = \frac{d}{dt}(z^2)$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$\frac{2x}{2z}\frac{dx}{dt} = \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{x}{z}\cdot\frac{dx}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dz}{dt} = \frac{4}{z}\cdot 500 = \frac{2000}{z}$$

$$= \frac{2000}{5}$$

Find z by plugging $x=4$ into

$$x^2+9=z^2$$

$$4^2+9=z^2$$

$$25=z^2$$

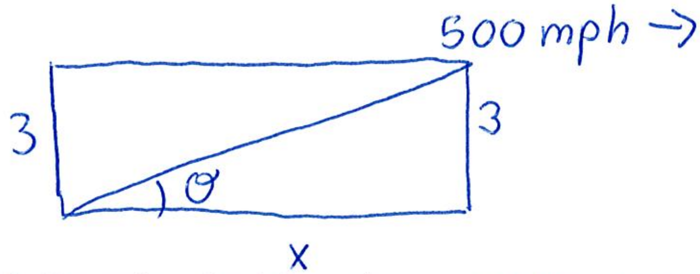
$$z=5$$

So $\frac{dz}{dt} = 400$

Example 4: **A plane is flying directly away from you at 500 mph at an altitude of 3 miles.**

b) How fast is the angle of elevation changing when it is $\pi/3$?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500 \text{ mph}$ **WANT:** $\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3}$

Step 3: Find an equation relating the relevant variables.

$$\tan\theta = \frac{3}{x} \iff \tan\theta = 3x^{-1}$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(3x^{-1})$$

$$\sec^2\theta \frac{d\theta}{dt} = -3x^{-2}\frac{dx}{dt}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{-3}{x^2}\frac{dx}{dt}$$

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{-3}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-3}{x^2}\cdot\cos^2\theta\cdot\frac{dx}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{d\theta}{dt} = \frac{-3}{x^2}\cdot\left(\cos\left(\frac{\pi}{3}\right)\right)^2\cdot 500$$

$$= \frac{-3}{x^2}\cdot\left(\frac{1}{2}\right)^2\cdot 500$$

$$= \frac{-375}{x^2}$$

$$= -\frac{375}{3} = -125$$

Find x by plugging $\theta = \pi/3$ into

$$\tan\theta = \frac{3}{x}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{3}{x}$$

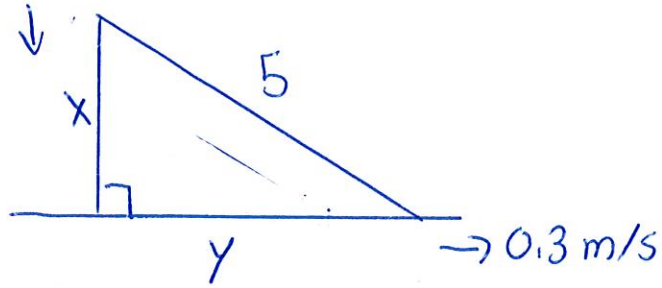
$$\frac{\sqrt{3}}{1} = \frac{3}{x}$$

$$\sqrt{3}\,x = 3$$

$$x = \frac{3}{\sqrt{3}} \Rightarrow x^2 = \frac{9}{3} = 3$$

Example 5: A ladder 5 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.3 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 3 m from the wall?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dy}{dt} = 0.3\frac{m}{s}$ **WANT:** $\left.\frac{dx}{dt}\right|_{y=3}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + y^2 = 5^2 \iff x^2 + y^2 = 25$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(25)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\frac{1}{2x}\cdot 2x\frac{dx}{dt} = \frac{1}{2x}\cdot(-2y)\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dx}{dt} = \frac{-3}{x}\cdot 0.3 = \frac{-0.9}{x}$$

Find x by plugging $y=3$ into

$$x^2+y^2=25$$

$$x^2+3^2=25$$

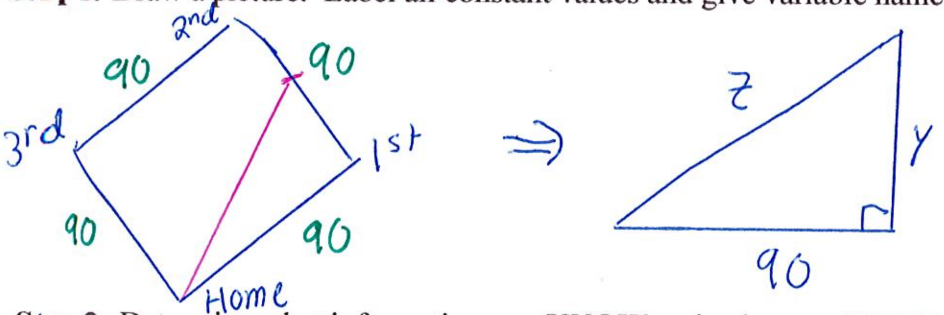
$$x^2 = 25-9 = 16$$

$$x = 4$$

$$\frac{dx}{dt} = \frac{-0.9}{4} = -0.225$$

Example 6: A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 14 ft/sec. At what rate is the player's distance from home base increasing when he is halfway from first to second base?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dy}{dt} = 14 \frac{ft}{s}$

WANT: $\left.\frac{dz}{dt}\right|_{y = \frac{90}{2} = 45 ft}$

Step 3: Find an equation relating the relevant variables.

$$90^2 + y^2 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(90^2 + y^2) = \frac{d}{dt}(z^2)$$

$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\frac{2y}{2z}\frac{dy}{dt} = \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{y}{z}\frac{dy}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dz}{dt} = \frac{45}{z} \cdot 14 = \frac{45 \cdot 14}{z}$$

$$\frac{dz}{dt} = \frac{45 \cdot 14}{45 \cdot \sqrt{5}} = \frac{14}{\sqrt{5}}$$

Find z by plugging $y = 45$ into

$$90^2 + y^2 = z^2$$

$$(2 \cdot 45)^2 + 45^2 = z^2$$

$$2^2 \cdot 45^2 + 45^2 = z^2$$

$$45^2(2^2 + 1) = z^2$$

$$45^2 \cdot 5 = z^2$$

$$z = 45\sqrt{5}$$