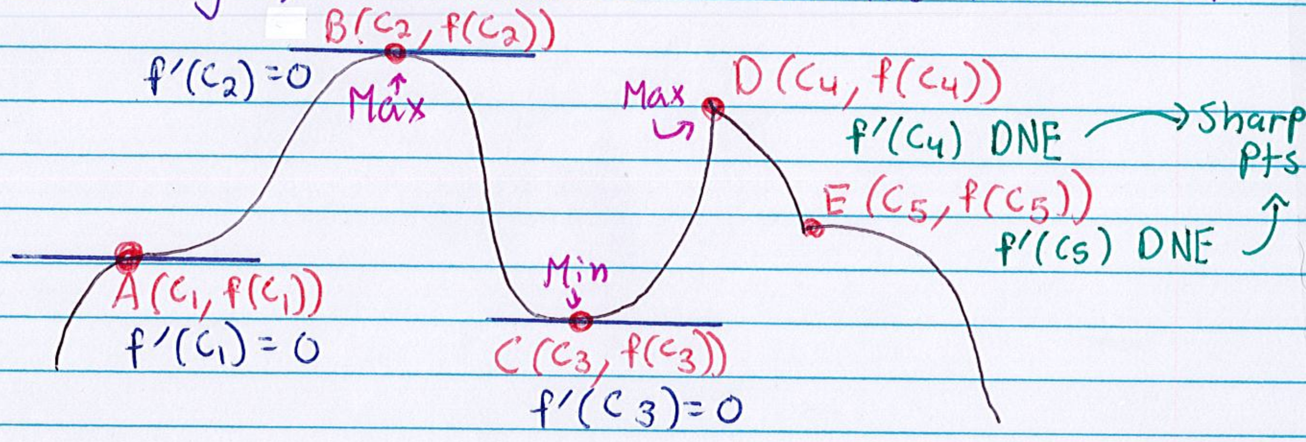
Lesson 13: Relative Extrema & Critical Numbers

Definition: ⓐ If $f(c) \geq f(x)$ for all x in an open interval I containing c, then $f(c)$ is a relative maximum.

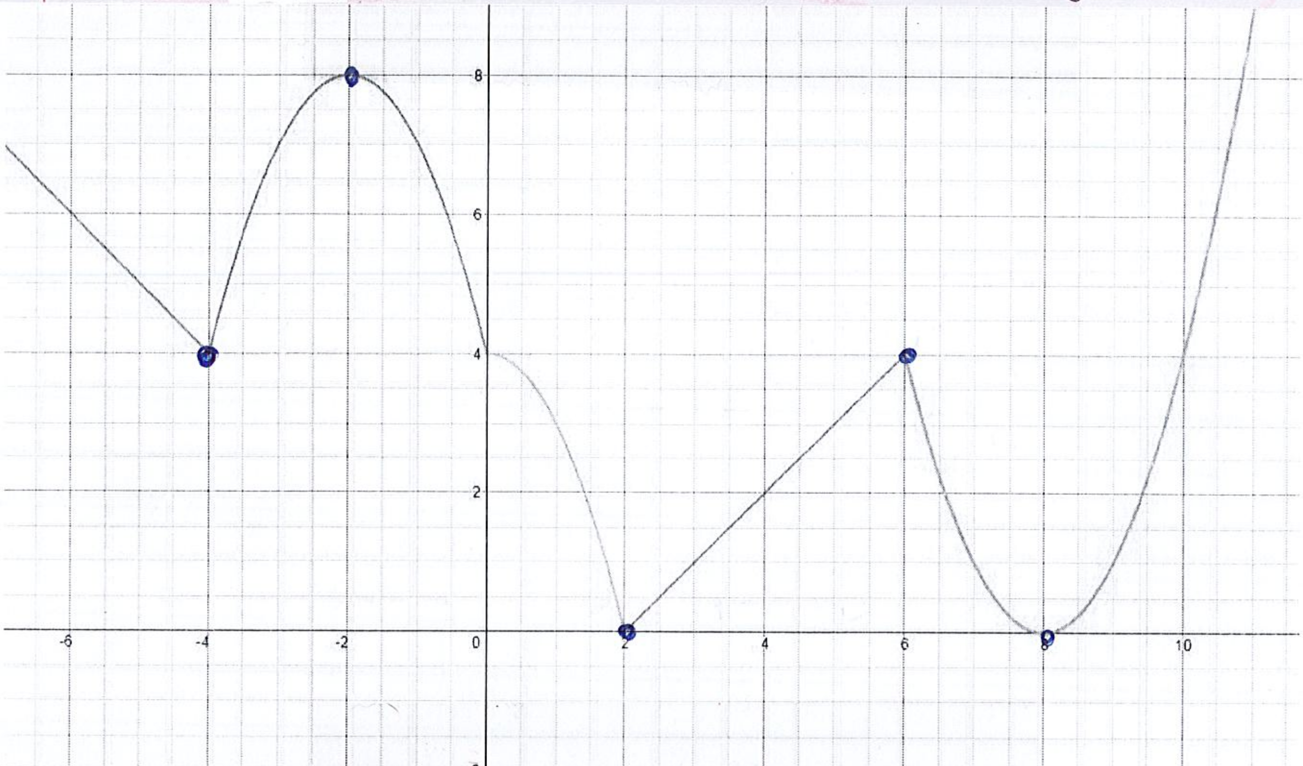
ⓑ If $f(c) \leq f(x)$ for all x in an open interval I containing c, then $f(c)$ is a relative minimum.



Overall all relative extremas occur at points where the derivative is zero or DNE.

Question: Is the reverse true?
i.e. If the derivative is zero or DNE, do we have a relative extrema?
No. Refer back to Points A and E in the picture above.

<u>Example 1</u>: Find the relative extrema of the graph below, and choose the correct statement regarding $f'(x)$.



We have a relative min at

- $(-4,4) \Rightarrow$ Sharp Point $\Rightarrow f'(x)$ @ $(-4,4)$ DNE
- $(2,0) \Rightarrow$ Sharp Point $\Rightarrow f'(x)$ @ $(2,0)$ DNE
- $(8,0) \Rightarrow$ Horizontal Tangent $\Rightarrow f'(x)=0$ @ $(8,0)$

We have a relative max at

- $(-2,8) \Rightarrow$ Horizontal Tangent $\Rightarrow f'(x)=0$ @ $(-2,8)$
- $(6,4) \Rightarrow$ Sharp Point $\Rightarrow f'(x)$ @ $(6,4)$ DNE

<u>Definition</u>: Let c be a # in the domain of f. If $f'(x)=0$ or $f'(x)$ DNE @ $x=c$, then c is a critical number.

<u>Example 1</u>: Find the critical numbers of the following functions:

ⓐ $y = x^4 - 2x^3$

i.e. Solve $y'(x)=0$ for x.

$y' = 4x^3 - 6x^2 = 0$

$$2x^2(2x-3)=0$$

$2x^2=0$	$2x-3=0$
$x=0$	$x=3/2$

Hence $x=0$, $x=3/2$ are the critical numbers.

ⓑ $y = 3 + 8x - \dfrac{8x^3}{3}$

i.e. Solve $y'(x)=0$ for x.

$$y' = 8 - 8x^2 = 0$$
$$8(1-x^2)=0$$
$$8(1-x)(1+x)=0$$

$1-x=0$	$1+x=0$
$x=1$	$x=-1$

Hence $x=-1$, $x=1$ are the critical numbers.

ⓒ $y = \dfrac{9x^2+13}{8x}$

i.e. Solve $y'(x)=0$ or DNE for x.

Let $u(x) = 9x^2+13$ $\quad v(x)=8x$

$u'(x) = 18x$ $\quad v'(x) = 8$

By Quotient Rule,

$$y' = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$
$$= \frac{(18x)(8x) - (9x^2+13)(8)}{(8x)^2}$$
$$= \frac{144x^2 - (72x^2+104)}{64x^2}$$
$$= \frac{144x^2 - 72x^2 - 104}{64x^2}$$
$$= \frac{72x^2-104}{64x^2}$$

So $y'=0$ when $72x^2-104=0 \Rightarrow x^2 = \dfrac{104}{72} = \dfrac{13}{9}$

$\Rightarrow x = \pm\sqrt{13/9}$

y' DNE when $64x^2=0 \Rightarrow x=0$

However $x = \pm\sqrt{13/9}$ are the only critical numbers because y was originally undefined at $x=0$.

(d) $y = 3x^4 e^x$

i.e. Solve $y'(x) = 0$ for x.

Let $u(x) = 3x^4$ $\quad v(x) = e^x$

$u'(x) = 12x^3$ $\quad v'(x) = e^x$

By Product Rule,

$$y' = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= 12x^3 e^x + 3x^4 e^x$$

$$y' = 12x^3 e^x + 3x^4 e^x = 0$$
$$3x^3 e^x (4+x) = 0$$

$3x^3 = 0$	$e^x = 0$	$4 + x = 0$
$x = 0$	↑ Never	$x = -4$

(e) $y = 10x^4 e^{5x+4}$

i.e. Solve $y'(x) = 0$ for x.

· Let $u(x) = 10x^4$ $\quad v(x) = e^{5x+4}$

$u'(x) = 40x^3$ $\quad v'(x) = 5e^{5x+4}$

By Product Rule,

$$y' = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= 40x^3 e^{5x+4} + (10x^4)(5e^{5x+4})$$
$$= 10x^3 e^{5x+4}(4+5x)$$

$$y' = 10x^3 e^{5x+4}(4+5x) = 0$$

$10x^3 = 0$	$e^{5x+4} = 0$	$4 + 5x = 0$
$x = 0$	↑ Never	$5x = -4$
		$x = -4/5$

Moreover $\exp[\text{any function of } x] \neq 0$ for all x.

(f) $y = 8x^2 - \dfrac{5}{x^2}$

i.e. Solve $y'(x) = 0$ for x.

Rewrite $y = 8x^2 - 5x^{-2}$

$$y' = 16x + 10x^{-3} = 0$$

$$16x + \frac{10}{x^3} = 0$$

$$\frac{16x}{1} = -\frac{10}{x^3}$$

$$16x^4 = -10$$

$$x^4 = -\frac{10}{16} = \frac{-5}{8}$$

$$x = \sqrt[4]{\frac{-5}{8}}$$ ⇒ But that's not possible.

Hence there is no critical point.

Example 2: Is $\pi/8$ a critical number for $y = 2\sin(2x) - 2\sqrt{2}\,x$?

i.e. Check $y'(\pi/8) = 0$.

$$y' = 2\cos(2x) \cdot 2 - 2\sqrt{2}$$
$$= 4\cos(2x) - 2\sqrt{2}$$

$$y'\left(\frac{\pi}{8}\right) = 4\cos\left(\frac{2\pi}{8}\right) - 2\sqrt{2}$$
$$= 4\cos\left(\frac{\pi}{4}\right) - 2\sqrt{2}$$
$$= 4\frac{\sqrt{2}}{2} - 2\sqrt{2}$$
$$= 2\sqrt{2} - 2\sqrt{2} = 0 \quad ✓$$

Yes $\pi/8$ is a critical number for y.