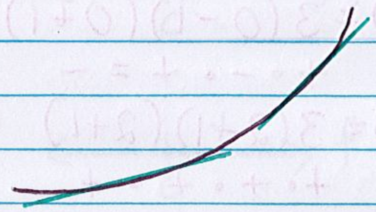
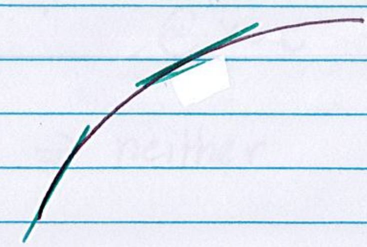
Lesson 14: Increasing and Decreasing Functions, and the First Derivative Test

A function is increasing if the function value gets bigger and bigger.

A function is decreasing if the function value gets smaller and smaller.





Notice that slope of the tangent lines when
- $f(x)$ increasing $\Rightarrow$ positive slopes
- $f(x)$ decreasing $\Rightarrow$ negative slopes

Theorem: Let $f(x)$ be a continuous and differentiable function on an open interval, I.
- If $f'(x) > 0$ for all x in I, then $f(x)$ increasing on I.
- If $f'(x) < 0$ for all x in I, then $f(x)$ decreasing on I.

Game Plan for Determining Increasing/Decreasing Intervals

① Find when $f'(x) = 0$.
② Draw a # line with the points from ①.
③ Determine test points using ②,
④ Plug those values into f' to determine whether it's positive or negative.
⑤ Use definition of increasing/decreasing

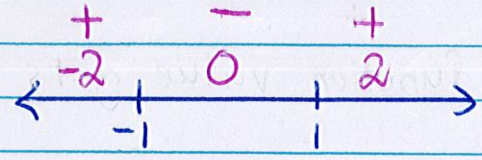
Example 1: Determine where each $f(x)$ is increasing/decreasing

ⓐ $f(x) = x^3 - 3x$

First find when $f'(x) = 0$

$$f'(x) = 3x^2 - 3 = 0$$
$$3(x^2 - 1) = 0$$
$$3(x-1)(x+1) = 0$$
$$x = \pm 1$$

Next draw a # line with $x=-1$, and $x=1$. With that number line determine test points and plug them into f' to determine whether f' is positive or negative in that interval.



$f'(x) = 3(x-1)(x+1)$

$f'(-2) = 3(-2-1)(-2+1)$

$+ \cdot - \cdot - = +$

$f'(0) = 3(0-1)(0+1)$

$+ \cdot - \cdot + = -$

$f'(2) = 3(2-1)(2+1)$

$+ \cdot + \cdot + = +$

Hence $f(x)$ is increasing at $(-\infty,-1)\cup(1,\infty)$
and decreasing at $(-1,1)$

(b) $f(x) = -2x^2e^{4x+1}$

First find when $f'(x)=0$.

Let, $u(x) = -2x^2 \qquad v(x) = e^{4x+1}$

$u'(x) = -4x \qquad v'(x) = 4e^{4x+1}$

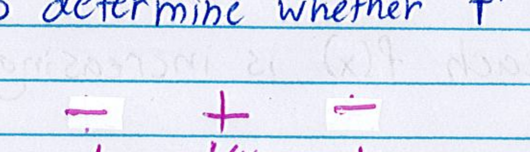
$f'(x) = u'(x)v(x) + u(x)v'(x)$

$= -4xe^{4x+1} + (-2x^2)\cdot 4e^{4x+1} = 0$

$-4xe^{4x+1}(1+2x) = 0$

$-4x=0$	$e^{4x+1}=0$	$1+2x=0$
$x=0$	↑	$x=-1/2$
	Never	

Next draw a # line with $x=0$, and $x=-1/2$. With that number line determine test points and plug them into f' to determine whether f' is positive or negative in that interval.



Note $e^x>0$ for all x

$f'(-1) = + \cdot + \cdot - = -$

$f'(-1/4) = + \cdot + \cdot + = +$

$f'(1) = \ \cdot + \cdot + = -$

Hence $f(x)$ is increasing at $(-1/2, 0)$
and decreasing at $(-\infty,-1/2)\cup(0,\infty)$

The First Derivative Test

Let c be a critical # of $f(x)$ that is continuous on an open interval, I, containing c.

① $\overset{+}{\longleftarrow}\underset{c}{|}\overset{-}{\longrightarrow} f'$ ex. $\cap$ $\Rightarrow$ relative max @ $x=c$

② $\overset{-}{\longleftarrow}\underset{c}{|}\overset{+}{\longrightarrow} f'$ ex. $\cup$ $\Rightarrow$ relative min @ $x=c$

③ $\overset{+}{\longleftarrow}\underset{c}{|}\overset{+}{\longrightarrow} f'$ ex. $\Rightarrow$ neither

④ $\overset{-}{\longleftarrow}\underset{c}{|}\overset{-}{\longrightarrow} f'$ ex. $\Rightarrow$ neither

Example 2: Given $f(x) = 2x^4 - 2x^3$

ⓐ Find where f is increasing/decreasing.

First find where $f'(x) = 0$.

$$f'(x) = 8x^3 - 6x^2 = 0$$
$$2x^2(4x-3) = 0$$
$$2x^2 = 0 \quad \Big| \quad 4x - 3 = 0$$
$$x = 0 \quad \Big| \quad x = 3/4$$

Next draw a # line with $x=0$, and $x=3/4$. With that number line determine test points and plug them into f' to determine whether f' is positive or negative in that interval.

$$\begin{array}{ccccc} - & & - & & + \\ -1 & & 1/2 & & 1 \\ \longleftarrow & | & & | & \longrightarrow f' \\ & 0 & & 3/4 & \end{array}$$

Note $x^2 > 0$ for all x.

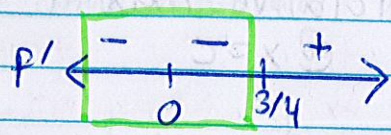
$f'(-1) = + \cdot - = -$

$f'(1/2) = + \cdot - = -$

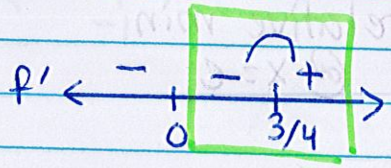
$f'(1) = + \cdot + = +$

Hence $f(x)$ is increasing at $(3/4, \infty)$ and decreasing at $(-\infty, 3/4)$

(b) Find the relative extrema of $f(x)$.
Using the # line, found in (a), and the First Derivative Test.



By Case 4, $x=0$ is not a relative extrema



By Case 2, $x=3/4$ is a relative min.

Example 3: Find the relative extrema for $g(x)$ when given it's derivative.

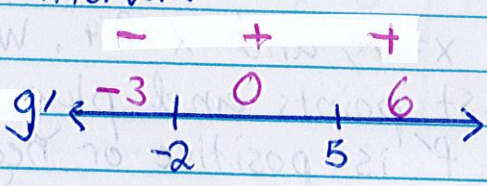
(a) $g'(x)=(3x+6)(x-5)^2$

Note we just need to set the derivative to 0, and solve.

$$g'(x)=(3x+6)(x-5)^2=0$$

$3x+6=0$	$(x-5)^2=0$
$x=-2$	$x-5=0$
	$x=5$

Next draw a # line with $x=-2$ and $x=5$. With that number line determine test points and plug them into f' to determine whether f' is positive or negative in that interval.



Note $(x-5)^2>0$ for all x.

$g'(-3)=-\cdot+=-$

$g'(0)=+\cdot+=+$

$g'(6)=+\cdot+=+$

Using the # line and the First Derivative Test,

- By Case 2, $x=-2$ is a relative min.
- By Case 3, $x=5$ is not a relative extrema.

(b) $g'(x)=e^{2x}(9-3x)$

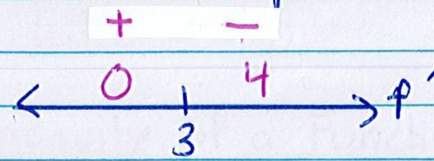
Note we just need to set the derivative to 0, and solve.

$$g'(x)=e^{2x}(9-3x)=0$$

Remember $e^{2x} \neq 0$. So $g'(x) = 0$ when

$$9 - 3x = 0$$
$$x = 3$$

Next draw a # line with $x = 3$. With that number line determine test points and plug them into f' to determine whether f' is positive or negative in that interval.



Note, $e^{2x} > 0$ for all x.

$g'(0) = + \cdot + = +$

$g'(4) = + \cdot - = -$

Using the # line and the First Derivative Test,

- By Case 1, $x = 3$ is a relative max.