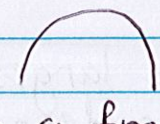
Lesson 15: Concavity, Inflection Pts, and the Second Derivative Test

Concave Up | Concave Down


Like a cup | Like a frown

Concavity of a Function: Suppose $f''(x)$ exists on an open interval, I, Then.
ⓐ If $f''(x) > 0$ for all x in I, then $f(x)$ is concave up on I.
ⓑ If $f''(x) < 0$ for all x in I, then $f(x)$ is concave down on I.

Game Plan for Determining Concavity
① Find when $f''(x) = 0$
② Draw a # line with the points from ①.
③ Determine test points using ②.
④ Plug those values into f'' to determine whether it's positive or negative
⑤ Use definition of concavity

Example 1: Determine the largest open interval(s) on which $f(x) = x^3 - x$ is concave up or down.
First find when $f''(x) = 0$

$$f'(x) = 3x^2 - 1 = 0$$
$$6x = 0$$
$$x = 0$$

Next draw a # line with $x=0$. With that number line determine test points and plug them into f'' to determine whether f'' is positive or negative in that interval.

f'': number line with 0 marked; test points -1 (−) and 1 (+)

$$f''(x) = 6x$$
$$f''(-1) = -6 < 0 \Rightarrow -$$
$$f''(1) = 6 > 0 \Rightarrow +$$

Hence $f(x)$ is concave up at $(0,\infty)$
and concave down at $(-\infty,0)$

<u>Example 2:</u> Let $f(x) = \frac{1}{12}x^4 - \frac{1}{3}x^3$

ⓐ Determine the largest open interval(s) on which $f(x)$ is increasing or decreasing.

First find when $f'(x) = 0$.

$$f'(x) = \frac{4}{12}x^3 - \frac{3}{3}x^2 = 0$$

$$\frac{1}{3}x^3 - x^2 = 0$$

$$3\left(\frac{1}{3}x^3 - x^2\right) = 0$$

$$x^3 - 3x^2 = 0$$

$$x^2(x-3) = 0$$

$$x^2 = 0 \quad \Big| \quad x - 3 = 0$$

$$x = 0 \quad \Big| \quad x = 3$$

Next draw a # line with $x=0$ and $x=3$. With that number line determine test points and plug them into f' to determine whether f' is positive or negative in that interval.

$-$ $-$ $+$

f' ← -1 | 1 | 4 →

0 3

Note $x^2 > 0$ for all x.

$f'(x) = x^2(x-3)$

$f'(-1) = + \cdot (-1-3)$

$+ \cdot - = -$

$f'(1) = + \cdot (1-3)$

$+ \cdot - = -$

$f(4) = + \cdot (4-3)$

$+ \cdot + = +$

<u>Increasing:</u> $(3,\infty)$

<u>Decreasing:</u> $(-\infty,3)$

ⓑ Determine the largest open interval(s) on which $f(x)$ is concave up or down.

First find when $f''(x) = 0$.

$$f'(x) = \frac{1}{3}x^3 - x^2$$

$$f''(x) = \frac{3}{3}x^2 - 2x = 0$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$x = 0$ | $x - 2 = 0$, $x = 2$

Next draw a # line with $x=0$ and $x=2$. With that number line determine test points and plug them into f'' to determine whether f'' is positive or negative in that interval.

f'': $+$ at -1 | 0 | $-$ at 1 | 2 | $+$ at 3

$f''(x) = x(x-2)$

$f''(-1) = -1(-1-2)$

$- \cdot - = +$

$f''(1) = 1(1-2)$

$+ \cdot - = -$

$f''(3) = 3(3-2)$

$+ \cdot + = +$

Hence $f(x)$ is concave up at $(-\infty, 0) \cup (2, \infty)$ and concave down at $(0, 2)$

ⓒ Determine the largest open interval(s) on which $f(x)$ is decreasing and concave up.

Using the # lines from ⓐ and ⓑ, and see where they overlap.

f': $-$ | 0 | $-$ | 3 | $+$

f'': $+$ | 0 | $-$ | 2 | $+$

	$(-\infty,0)$	$(0,2)$	$(2,3)$	$(3,\infty)$
f'	$-$	$-$	$-$	$+$
f''	$+$	$-$	$+$	$+$

Hence answer: $(-\infty, 0) \cup (2, 3)$

To understand more about the shape of the graph of a function, we need to introduce inflection points.

<u>Definition:</u> Inflection Points exist when the concavity changes from positive to negative and vice versa.

<u>Steps in Finding the Inflection Pts.</u>

① Find the points on the curve where the second derivative is 0 or DNE.

<u>i.e.</u> Find where $f''(x)=0$ or $f''(x)$ DNE.

② Test whether the concavity changes at these points.

<u>i.e.</u> $f'' \longleftarrow - \underset{c}{|} + \longrightarrow$ or $f'' \longleftarrow + \underset{c}{|} - \longrightarrow$

where c is a point found in ①.

<u>Example 3:</u> Find the inflection point(s) of $f(x) = 9\ln(x^2+1)$ if they exist.

First find where $f''(x)=0$ or DNE.

$$f'(x) = 9 \cdot \frac{1}{x^2+1} \cdot 2x = \frac{18x}{x^2+1}$$

Let $u(x) = 18x \qquad v(x) = x^2+1$

$u'(x) = 18 \qquad v'(x) = 2x$

$$f''(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v^2(x)}$$

$$= \frac{18(x^2+1) - 18x(2x)}{(x^2+1)^2}$$

$$= \frac{18x^2 + 18 - 36x^2}{(x^2+1)^2}$$

$$= \frac{18 - 18x^2}{(x^2+1)^2}$$

$$= \frac{18(1-x^2)}{(x^2+1)^2}$$

$$= \frac{18(1-x)(1+x)}{(x^2+1)^2}$$

So $f''(x)=0 \Rightarrow 18(1-x)(1+x)=0 \Rightarrow x=\pm 1$

$f''(x)$ DNE $\Rightarrow (x^2+1)^2=0 \Rightarrow x^2+1=0$

$x^2=-1$

$x=\pm\cancel{\sqrt{-1}}$

Next draw a # line with $x=-1$ and $x=1$. With that number line determine test points and plug them into f'' to determine whether f'' is positive or negative in that interval.

$$\begin{array}{cccccc} & - & & + & & - \\ f'' & \leftarrow -2 & \mid & 0 & \mid & 2 \rightarrow \\ & & -1 & & 1 & \end{array}$$

Note $(x^2+1)^2>0 \Rightarrow x>0$

$f''(-2)=\dfrac{18(1-(-2))(1-2)}{+} \Rightarrow \dfrac{+\cdot+\cdot-}{+}=-$

$f''(0)=\dfrac{18(1-0)(1+0)}{+} \Rightarrow \dfrac{+\cdot+\cdot+}{+}=+$

$f''(2)=\dfrac{18(1-2)(1+2)}{+} \Rightarrow \dfrac{+\cdot-\cdot+}{+}=-$

Note we are looking for sign change:

So from the # line, we see a change at $x=-1$ and $x=1$.

Second Derivative Test

Let $f(x)$ be a function such that $f'(c)=0$ and $f''(x)$ exists on an open interval containing c.

① If $f''(c)>0$ then $f(x)$ has a relative min at $x=c$.

② If $f''(c)<0$ then $f(x)$ has a relative max at $x=c$.

If $f''(c)=0$, the Second Derivative Test does not apply. It does not necessarily mean that we have a relative extrema at these points. It just means you need to apply First Derivative Test.

How to Apply the Second Derivative to Find Relative Extrema

① Find where $f'(x)=0$

② Find $f''(x)$

③ Plug ① into ②.

④ Use the Second Derivative

↳ If it fails apply the First Derivative Test,

Example 4; Find all the relative extrema of the following functions, if they exists.

ⓐ $f(x) = x^4 - 8x^2$

First find when $f'(x) = 0$

$f'(x) = 4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x(x-2)(x+2) = 0$

$4x = 0$	$x - 2 = 0$	$x + 2 = 0$
$x = 0$	$x = 2$	$x = -2$

Next find $f''(x)$,

$f''(x) = 12x^2 - 16$

Now plug $x = -2, 0, 2$ into $f''(x)$,

$f''(-2) = 12(-2)^2 - 16 = 32$

$f''(0) = 12(0)^2 - 16 = -16$

$f''(2) = 12(2)^2 - 16 = 32$

By the Second Derivative Test, we have

a relative min @ $x = -2$ b/c $f''(-2) > 0$

a relative max @ $x = 0$ b/c $f''(0) < 0$

a relative min @ $x = 2$ b/c $f''(2) > 0$

ⓑ $f(x) = \frac{3}{5}x^5 - x^4$

First find when $f'(x) = 0$

$f'(x) = \frac{3}{5}(5)x^4 - 4x^3 = 0$

$3x^4 - 4x^3 = 0$

$x^3(3x - 4) = 0$

$x^3 = 0$	$3x - 4 = 0$
$x = 0$	$x = 4/3$

Next find $f''(x)$.

$$f'(x) = 3x^4 - 4x^3$$
$$f''(x) = 12x^3 - 12x^2$$
$$= 12x^2(x-1)$$

Now plug $x = 0, 4/3$ into $f''(x)$.

$$f''(0) = 12(0)^2(0-1) = 0$$
$$f''(4/3) = 12\left(\frac{4}{3}\right)^2\left(\frac{4}{3} - 1\right) = \frac{64}{9}$$

By the Second Derivative Test, we have a relative minimum at $x = 4/3$ b/c $f''(4/3) = 64/9 > 0$

☆ Note that $f''(0) = 0$ which means that we can't apply the Second Derivative Test. Hence we need to apply the First Derivative Test for $x = 0$.

To apply the First Derivative Test, we need to create a # for $x=0$ with $f'(x) = 3x^4 - 4x^3 = x^3(3x-4)$

$+ \quad -$

$-1 \quad 1$

f' ←——+——→

0

$$f'(-1) = (-1)^3(3(-1)-4)$$
$$- \cdot - = +$$
$$f'(1) = 1^3(3-4)$$
$$+ \cdot - = -$$

Hence by the First Derivative Test, we have a relative max at $x = 0$