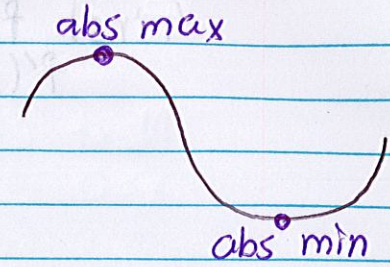
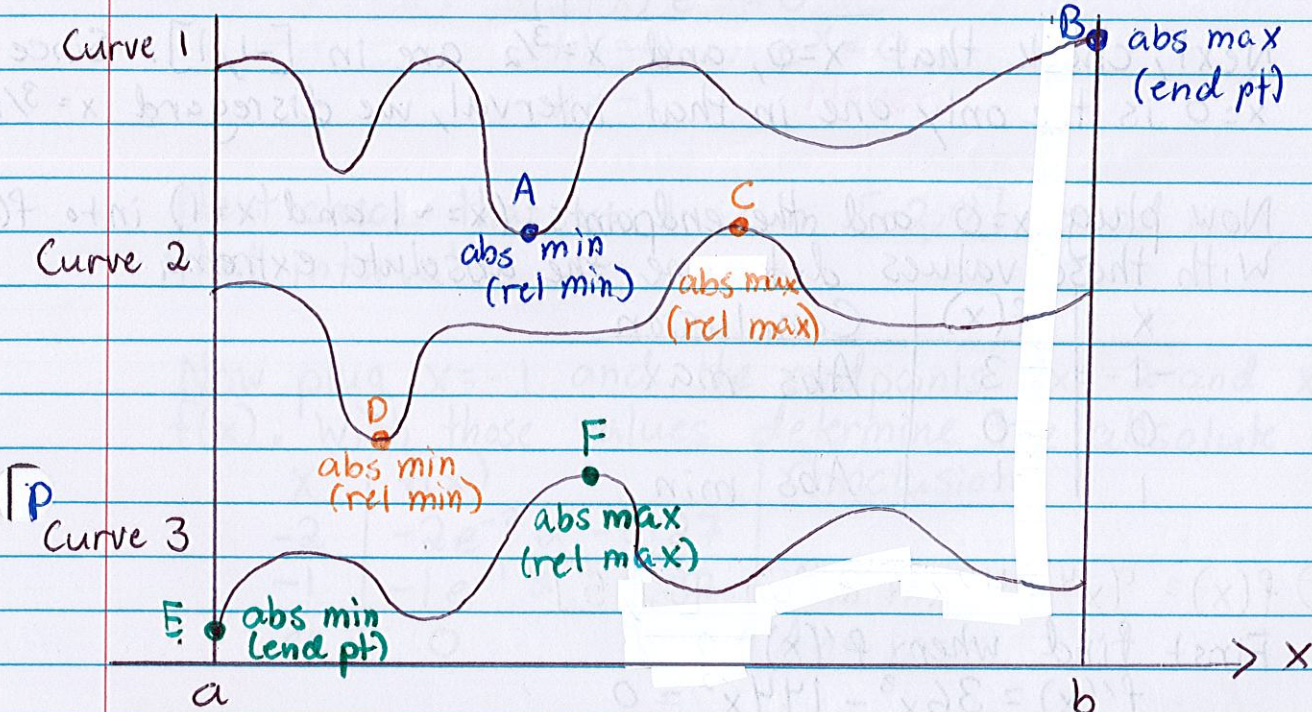
Lesson 16: Absolute Extrema on an Interval

Definition: • An absolute max is the largest function value on the entire interval.

• An absolute min is the smallest function value on the entire interval.





Theorem: If $f(x)$ is continuous on a closed interval $[a,b]$ then $f(x)$ has both an absolute max and min on the interval.

Note the absolute extrema only occur either at
- critical numbers, or
- endpoints

Steps to Find the Absolute Extrema

① Find all critical numbers. i.e. $f'(x)=0$

② Check if points from ① are in the interval given.

③ Plug points from ② and the included endpoints into $f(x)$.

④ Compare the function values

↳ Biggest $f(x)$ value in ② ⇒ abs max

↳ Smallest $f(x)$ value in ② ⇒ abs min

Example 1: Find the absolute extrema of

ⓐ $f(x) = x^4 - 2x^3$ on $[-1, 1]$

First find when $f'(x) = 0$.

$$f'(x) = 4x^3 - 6x^2 = 0$$
$$2x^2(2x-3) = 0$$
$$2x^2 = 0 \quad 2x - 3 = 0$$
$$x = 0 \quad x = 3/2$$

Next, check that $x=0$, and $x = 3/2$ are in $[-1, 1]$. Since $x=0$ is the only one in that interval, we disregard $x = 3/2$.

Now plug $x=0$ and the endpoints ($x=-1$ and $x=1$) into $f(x)$. With those values determine the absolute extrema.

x	f(x)	Conclusion
-1	3	Abs max
0	0	
1	-1	Abs min

ⓑ $f(x) = 9x^4 - 48x^3 + 2$ on $[-1, 5]$

First find when $f'(x) = 0$.

$$f'(x) = 36x^3 - 144x^2 = 0$$
$$36x^2(x-4) = 0$$
$$36x^2 = 0 \quad x - 4 = 0$$
$$x = 0 \quad x = 4$$

Next check that $x=0$, and $x=4$ are in $[-1, 5]$. Good news, both are in that interval.

Now plug $x=0$, $x=4$ and the endpoints ($x=-1$ and $x=5$) into $f(x)$. With those values determine the absolute extrema

x	f(x)	Conclusion
-1	59	Abs max
0	2	
4	-766	Abs min
5	-373	

ⓒ $f(x) = xe^x$ on $[-2, 0]$

First find when $f'(x) = 0$.

Let $u(x) = x$ $\quad v(x) = e^x$

$u'(x) = 1$ $\quad v'(x) = e^x$

By Product Rule,

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 1 \cdot e^x + x \cdot e^x = 0$$
$$(1+x)e^x = 0$$
$$1 + x = 0 \qquad e^x = 0$$
$$x = -1 \qquad \text{Never}$$

Next check that $x = -1$ is in $[-2, 0]$. Good news, it is in that interval.

Now plug $x = -1$ and the endpoints ($x = -2$ and $x = 0$) into $f(x)$. With those values determine the absolute extrema.

x	f(x)	Conclusion
-2	$-2e^{-2} \approx -0.27$	
-1	$-1e^{-1} \approx -0.37$	Abs min
0	0	Abs max

ⓓ $f(x) = \dfrac{6x^2}{x+1}$ on $(-1, 4]$

First find when $f'(x) = 0$.

Let $u(x) = 6x^2$ $\quad v(x) = x + 1$

$u'(x) = 12x$ $\quad v'(x) = 1$

By Quotient Rule,

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$
$$= \frac{12x(x+1) - 6x^2 \cdot 1}{(x+1)^2} = 0$$
$$= \frac{12x^2 + 12x - 6x^2}{(x+1)^2} = 0$$
$$\frac{6x^2 + 12x}{(x+1)^2} = 0$$

$$6x(x+2)=0$$
$$6x=0 \qquad x+2=0$$
$$x=0 \qquad x=-2$$

Next check that $x=0$ and $x=-2$ are in $(-1,4]$. Since $x=0$ is the only one in that interval, we disregard $x=-2$.

Now plug $x=0$ and the endpoint ($x=-4$) into $f(x)$. With those values determine the absolute extrema.

x	$f(x)$	Conclusion
0	0	Abs min
4	19.2	Abs max

ⓔ $f(x) = -x^2 - 2x$ on $(-2,0)$

First find when $f'(x)=0$.

$$f'(x) = -2x-2=0$$
$$-2(x+1)=0$$
$$x=-1$$

Next check that $x=-1$ is in $(-2,0)$. Good news, it is in that interval.

Note we only have one value to test. When you get one value, you need to use First/Second Derivative Test.

By the Second Derivative Test,

$$f''(x) = -2$$
$$f''(-1) = -2 < 0 \Rightarrow \text{relative max at } x=-1$$
$$\Rightarrow \text{absolute max at } x=-1$$