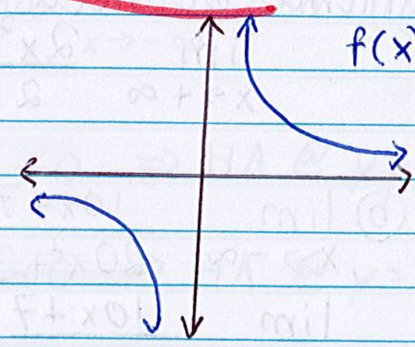
Lesson 18: Limits at Infinity

Recall

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^+} \frac{1}{x} = +\infty$$



Now we want

$$\lim_{x\to -\infty} \frac{1}{x} = 0 \qquad \lim_{x\to +\infty} \frac{1}{x} = 0$$

The purpose of revisiting this topic is to determine
① End Behavior
② If we have a Horizontal/Slant Asymptote.

Example 1: Find the following limits

ⓐ $\lim_{x\to+\infty} \frac{3}{x} = 3 \lim_{x\to+\infty} \frac{1}{x} = 3\cdot 0 = 0$

ⓑ $\lim_{x\to-\infty} \left(\frac{x}{3} + 2\right) = \frac{-\infty}{3} + 2 = -\infty + 2 = -\infty$

ⓒ $\lim_{x\to+\infty} \left(\frac{x}{2} + \frac{5}{x}\right) = \frac{+\infty}{2} + 5\cdot 0 = \frac{\infty}{2} = \infty$

General Rule: The limit of a rational function $f(x) = \frac{p(x)}{q(x)}$ as $x \to \pm\infty$ is determined by the leading terms of the numerator and the denominator.

Recall: A leading term of a polynomial is the term that has the highest power of x.

Example 2: Find the following limits:

ⓐ $\lim_{x\to+\infty} \frac{2x^2+1}{x^2-1}$

$\lim_{x\to+\infty} \frac{2x^2+1}{x^2-1} = "\frac{\infty}{\infty}" \Rightarrow$ No-no.

Hence we need to try using the General Rule,

$$\lim_{x \to +\infty} \frac{2x^2}{2} = \lim_{x \to +\infty} 2 = 2$$

ⓑ $\lim_{x \to -\infty} \frac{10x+7}{20x^2+3}$

$$\lim_{x \to -\infty} \frac{10x+7}{20x^2+3} = \frac{-\infty}{\infty} \Rightarrow \text{No-no}$$

Hence we need to try using the General Rule,

$$\lim_{x \to -\infty} \frac{10x}{20x^2} = \lim_{x \to -\infty} \frac{1}{2x} = \frac{1}{2} \cdot \lim_{x \to -\infty} \frac{1}{x} = \frac{1}{2} \cdot 0 = 0$$

ⓒ $\lim_{x \to +\infty} \frac{5x^3+9}{2x^2+1}$

$$\lim_{x \to +\infty} \frac{5x^3+9}{2x^2+1} = \frac{\infty}{\infty} \Rightarrow \text{No-no}$$

Hence we need to try using the General Rule,

$$\lim_{x \to +\infty} \frac{5x^3}{2x^2} = \lim_{x \to +\infty} \frac{5}{2} \cdot x = \frac{5}{2} \cdot \infty = \infty$$

Horizontal Asymptotes

The line $y = L$, where L is a constant, is a horizontal asymptote of $f(x)$ if

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to +\infty} f(x) = L$$

Note both limits need not match.
For example when $f(x) = e^x$

Example 3: Find, if they exist, the horizontal asymptote of the following functions:

ⓐ $h(x) = \frac{x-4}{x^2-1}$

To determine the HA we need to find $\lim_{x\to-\infty}$ and $\lim_{x\to+\infty}$ with the general rule.

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{x}{x^2} = \lim_{x\to-\infty} \frac{1}{x} = 0 \Rightarrow \text{HA @ } y=0$$

$$\lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} \frac{x}{x^2} = \lim_{x\to+\infty} \frac{1}{x} = 0 \Rightarrow \text{HA @ } y=0$$

ⓑ $h(x) = \frac{x^3+5}{2x+1}$

To determine the HA we need to find $\lim_{x\to-\infty}$ and $\lim_{x\to\infty}$ with the general rule.

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{x^3}{2x} = \lim_{x\to-\infty} \frac{x^2}{2} = \infty$$

$$\lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} \frac{x^3}{2x} = \lim_{x\to+\infty} \frac{x^2}{2} = \infty$$

Since each limit don't equal a constant, there are NO HA!

Slant Asymptotes

The line $y = ax+b$, where a and b are constants and $a \neq 0$, is a slant asymptote of $f(x)$ if $f(x)$ gets closer and closer to $y = ax+b$ as $x \to \pm\infty$

To find the slant asymptotes of a rational function, we use

- Synthetic Division
- Long Division