Lesson 19: A Summary of Curve Sketching

Now we are going to assemble what we learned to analyze and sketch rational functions.

Example 1: Analyze and sketch the graph of $f(x) = \frac{x^2 - x}{x - 3}$

ⓘ Domain of f: i.e. When does f(x) DNE?
Fractions are undefined when denominator $= 0$

$$x - 3 = 0$$
$$x = 3 \Rightarrow \text{Domain: } (-\infty, 3) \cup (3, \infty)$$

ⓘⓘ x-intercept: i.e. Set $y = 0$ and solve for x.

$$\frac{0}{1} = \frac{x^2 - x}{x - 3}$$
$$0(x-3) = x^2 - x$$
$$0 = x^2 - x$$
$$0 = x(x-1)$$
$$x = 0 \quad \Big| \quad x - 1 = 0, \; x = 1$$

x-intercepts: (0,0), (1,0)

ⓘⓘⓘ y-intercept: i.e. Set $x = 0$ and solve for y.

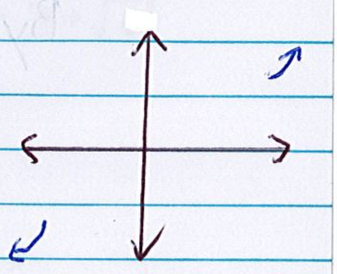
$$y = \frac{0^2 - 0}{0 - 3} = \frac{0}{-3} = 0$$

y-intercept: (0,0)

ⓘⓥ End Behavior: i.e. Use the General Rule for the following limits:

ⓐ $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2}{x} = \lim_{x \to -\infty} x = -\infty$

ⓑ $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2}{x} = \lim_{x \to +\infty} x = +\infty$



Note that $f(x) = \frac{x(x-1)}{x-3}$

Ⓥ Asymptote

ⓐ Vertical: ① Check that $f(x)$ is simplify ✓

② Set the denominator to 0, and solve.

$$x-3=0$$
$$x=3 \quad \Rightarrow \underline{VA}; \ x=3$$

ⓑ Horizontal; Use ④

Since $\lim\limits_{x \to \pm\infty} f(x) \neq L$, where L is a constant, then there is no HA.

ⓒ Slant: ① Check that the difference between the powers of the leading terms of numerator and denominator is equal to 1. ✓

② If so, use Synthetic Division or Long Division. If not, there is no Slant Asymptote.

Since ① checks, let's use Synthetic Division.

$$\begin{array}{r|rrr} 3 & 1 & -1 & 0 \\ & \downarrow & 3 & 6 \\ \hline & 1 & 2 & 6 \end{array} \quad \Rightarrow f(x) = \underset{\Downarrow}{\boxed{x+2}} + \frac{6}{x-3}$$

Slant Asymptote: $y = x+2$

Ⓥⓘ Critical #s; i.e. $f'(x)=0$ and $f'(x)$ DNE

Let $u(x) = x^2 - x \qquad v(x) = x-3$

$u'(x) = 2x-1 \qquad v'(x) = 1$

By Quotient Rule,

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{(2x-1)(x-3) - (x^2-x)\cdot 1}{(x-3)^2}$$

	$2x$	-1
x	$2x^2$	$-x$
-3	$-6x$	3

$$= \frac{2x^2-7x+3 - x^2 + x}{(x-3)^2}$$

$$= \frac{x^2-6x+3}{(x-3)^2} = 0$$

- $f'(x) = 0$ when $x^2 - 6x + 3 = 0$

By the Quadratic Formula,

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-12}}{2} = \frac{6 \pm \sqrt{24}}{2} = \frac{6 \pm 2\sqrt{6}}{2} = 3 \pm \sqrt{6}$$

- $f''(x)$ DNE when $(x-3)^2 = 0$

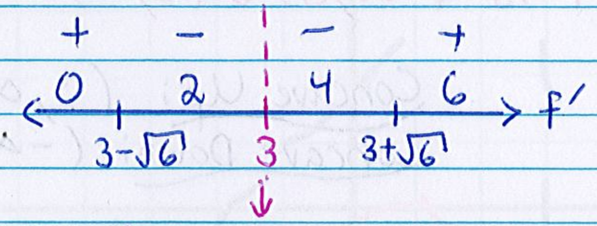
$$x = 3$$

Recall domain of f is $(-\infty, 3) \cup (3, \infty)$. So $x = 3$ can't be a critical #.

Critical #s: $x = 3 \pm \sqrt{6}$

ⓥⅱ Increasing / Decreasing:

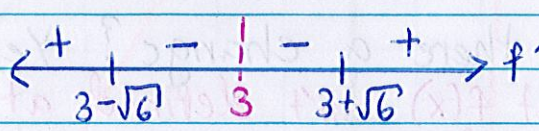
Note denominator is always positive.



Inc: $(-\infty, 3-\sqrt{6}) \cup (3+\sqrt{6}, \infty)$

Dec: $(3-\sqrt{6}, 3+\sqrt{6})$

ⓥⅲ Relative Extrema: Use ⓥⅱ and First Derivative Test.



Relative max: $x = 3 - \sqrt{6}$

Relative min: $x = 3 + \sqrt{6}$

ⓘⓧ Concavity: i.e. $f''(x) = 0$ and $f''(x)$ DNE

Let $u(x) = x^2 - 6x + 3$ $\qquad v(x) = (x-3)^2$

$u'(x) = 2x - 6$ $\qquad v'(x) = 2(x-3)$

$= 2(x-3)$

By Quotient Rule,

$$f''(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$f''(x) = \frac{2(x-3)(x-3)^2 - 2(x-3)(x^2-6x+3)}{(x-3)^4}$$

$$= \frac{2\cancel{(x-3)}\left[(x-3)^2 - (x^2-6x+3)\right]}{(x-3)^{\cancel{4}3}}$$

$$= \frac{2\left[\cancel{x^2} - \cancel{6x} + 9 - \cancel{x^2} + \cancel{6x} - 3\right]}{(x-3)^3}$$

$$= \frac{2 \cdot 6}{(x-3)^3}$$

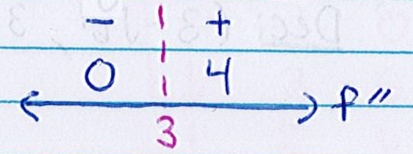
$$= \frac{12}{(x-3)^3} = 0$$

- $f''(x) = 0$ when $12 = 0$
  But that can never happen so $f''(x) \neq 0$

- $f''(x)$ DNE when $(x-3)^3 = 0$
  $x = 3$

Recall domain of f is $(-\infty, 3) \cup (3, \infty)$.

So

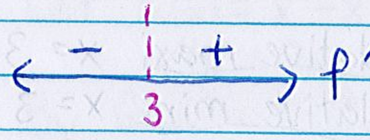


Concave Up: $(3, \infty)$

Concave Down: $(-\infty, 3)$

b/c f isn't defined at $x = 3$

ⓧ Inflection Point(s): Use ⓘⓧ and check for sign change.

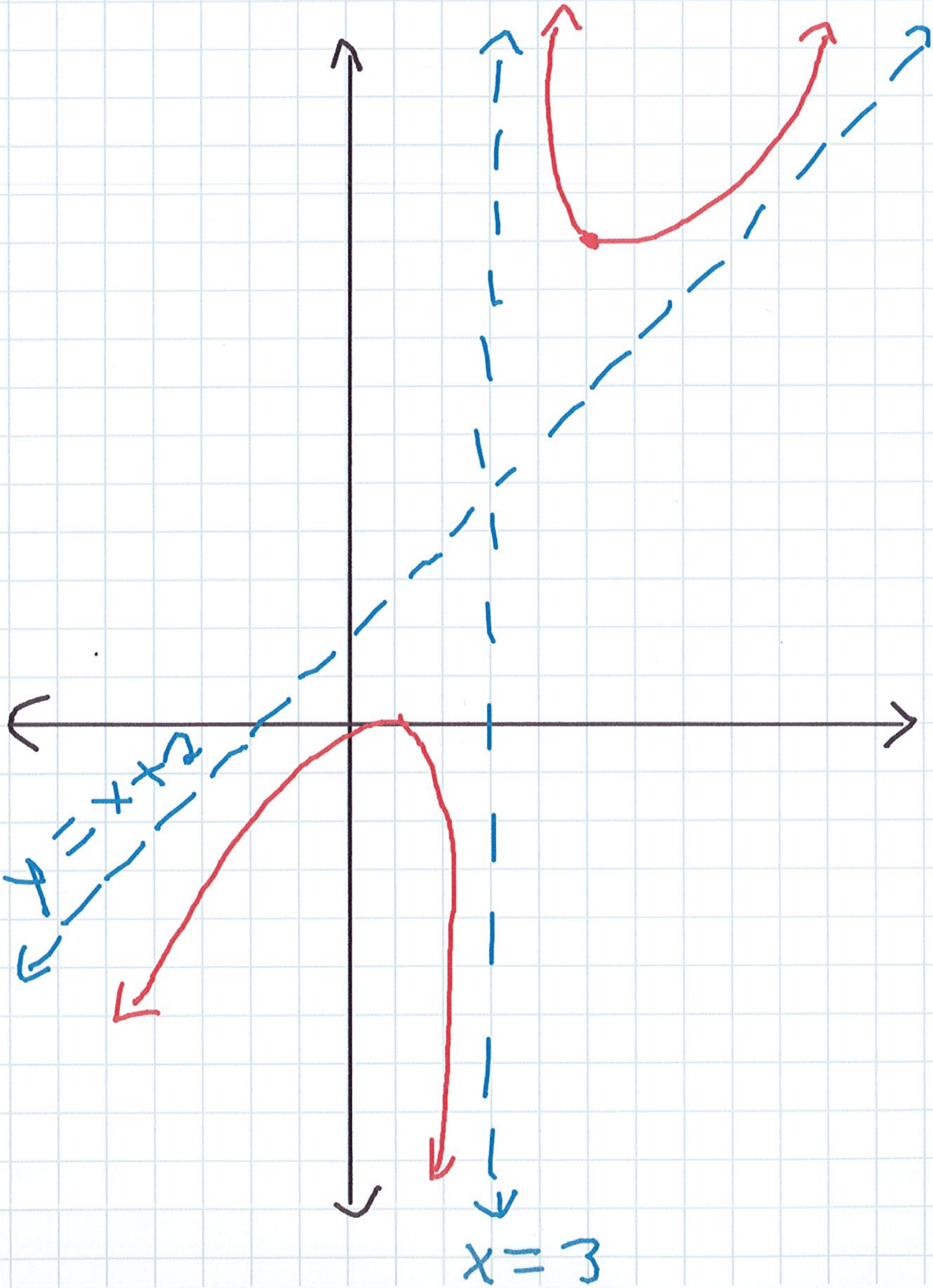


Is there a change? Yes

But f(x) isn't defined at x=3, so no inflection point.

ⓧi Graph

$$f(x) = \frac{x^2 - x}{x - 3}$$



(xi) Graph $f(x) = \frac{x^2 - x}{x - 3}$