Lesson 1: Finding Limits Numerically + One-Sided Limits

Definition: If $f(x)$ approaches $(\rightarrow)$ L as $x \rightarrow c$ we say that the limit of $f(x)$ as $x \rightarrow c$ is L.

i.e. $\lim\limits_{x \to c} f(x) = L$

Note that f does not need to be defined at $x = c$ for the limit to exist.

Definition: A one-sided limit is the value that the function $f(x) \rightarrow L$ as $x \rightarrow c$ from the left or right.

- Left-sided Limit: If $f(x) \rightarrow L$ as $x \rightarrow c$ from the left,
  $\lim\limits_{x \to c^-} f(x) = L$
- Right-sided Limit: If $f(x) \rightarrow L$ as $x \rightarrow c$ from the right
  $\lim\limits_{x \to c^+} f(x) = L$

Good news is if $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = L$, then $\lim\limits_{x \to c} f(x) = L$

But how do we find these limits?
We evaluate $f(x)$ at values of x that are getting closer and closer to c and see what happens with the values of the function.

Example 1: Evaluate numerically

ⓐ $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (2x-3)$

ⓑ $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} (2x-3)$

ⓒ $\lim\limits_{x \to 4} f(x) = \lim\limits_{x \to 4} (2x-3)$

Plug each # into $f(x) = 2x - 3$.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	4.8	4.98	4.998	–	5.002	5.02	5.2

ⓐ $\lim_{x \to 4^-} (2x-3) = 5$ b/c the left side is getting closer to 5.

ⓑ $\lim_{x \to 4^+} (2x-3) = 5$ b/c the right side is getting closer to 5.

ⓒ $\lim_{x \to 4} (2x-3) = 5$ b/c ⓐ and ⓑ are the same

But why can't I just plug in 4 into $f(x) = 2x - 3$?
The reason why is because the function may not be defined at that #.

Example 2: Evaluate numerically

ⓐ $\lim_{x \to 3^-} \frac{x^3 - 3x^2}{x-3}$ ⓑ $\lim_{x \to 3^+} \frac{x^3 - 3x^2}{x-3}$ ⓒ $\lim_{x \to 3} \frac{x^3 - 3x^2}{x-3}$

Plug each # into $f(x) = \frac{x^3 - 3x^2}{x-3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	8.41	8.9401	8.994	–	9.006	9.0601	9.6

ⓐ $\lim_{x \to 3^-} \frac{x^3 - 3x^2}{x-3} = 9$ b/c the left side is getting closer to 9.

ⓑ $\lim_{x \to 3^+} \frac{x^3 - 3x^2}{x-3} = 9$ b/c the right side is getting closer to 9,

ⓒ $\lim_{x \to 3} \frac{x^3 - 3x^2}{x-3} = 9$ b/c ⓐ and ⓑ are the same

Definition: If $f(x)$ increase or decrease without bound as $x \to c$ then $\lim_{x \to c} f(x)$ is an infinite limit.

- If $f(x)$ increases w/o bound, $\lim_{x \to c} f(x) = +\infty$

• If $f(x)$ decreases without bound,

$$\lim_{x \to c} f(x) = -\infty$$

Example 3: Evaluate numerically

ⓐ $\lim_{x \to 0^-} \frac{1}{x}$ ⓑ $\lim_{x \to 0^+} \frac{1}{x}$ ⓒ $\lim_{x \to 0} \frac{1}{x}$

Plug each # into $f(x) = 1/x$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-10	-100	-1000	$-$	1000	100	10

ⓐ $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ b/c the left side is getting bigger and bigger towards $-\infty$

ⓑ $\lim_{x \to 0^+} \frac{1}{x} = \infty$ b/c the right side is getting bigger and bigger towards ∞

ⓒ $\lim_{x \to 0} \frac{1}{x} = \text{DNE}$ b/c ⓐ and ⓑ don't match.

Finding Limits Graphically

Graphically, we will look at the portion of the curve of $f(x)$ near $x=c$ and see what the function value, y, approaches as x gets closer to c from the left or the right, respectively.

Again if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, then

$$\boxed{\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)} \quad (*)$$

Note that this doesn't imply that $(*) = f(c)$.

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Graphically, we will look at the portion of the curve of $f(x)$ near $x = c$ and see what the function value, y, approaches as x gets closer to c from the left or the right, respectively.

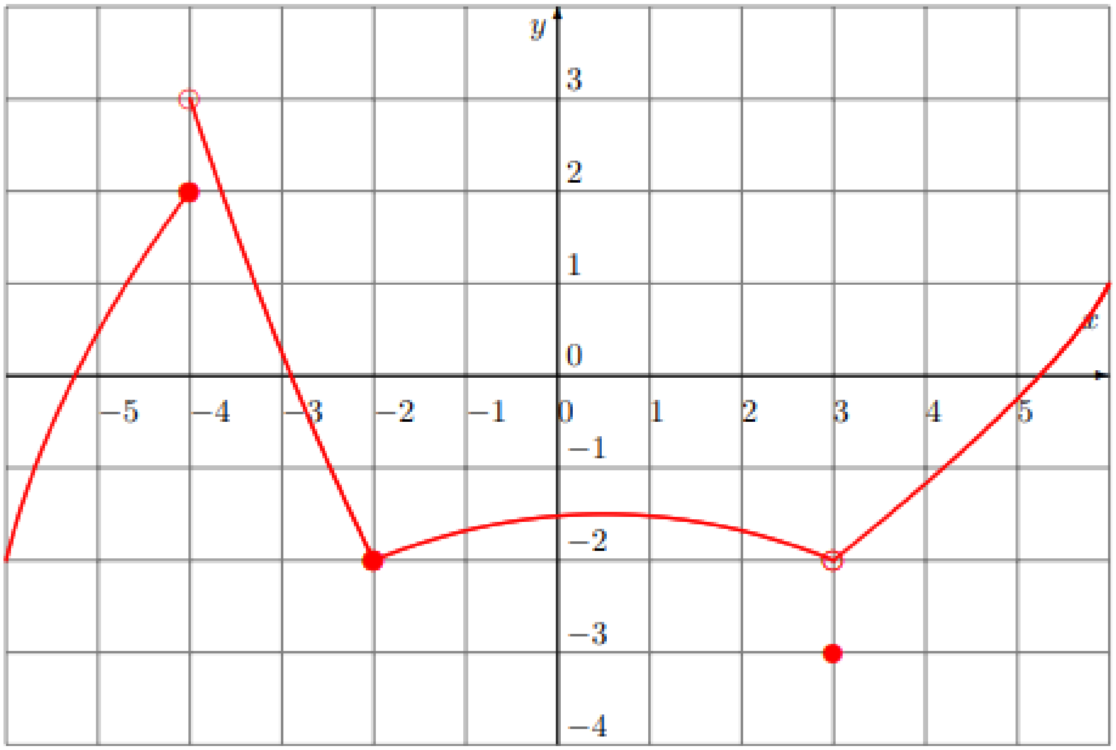
If $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$,

$$\boxed{\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)} \quad (*)$$

Note this doesn't imply that $(*) = f(c)$.

Example 3 (From Worksheet)

3. Consider the following function defined by its graph:



Find the following limits:

A) $\lim_{x\to -4^-} f(x) = 2$

E) $\lim_{x\to -2^-} f(x) = -2$

I) $\lim_{x\to 3^-} f(x) = -2$

B) $\lim_{x\to -4^+} f(x) = 3$

F) $\lim_{x\to -2^+} f(x) = -2$

J) $\lim_{x\to 3^+} f(x) = -2$

C) $\lim_{x\to -4} f(x) =$ DNE

G) $\lim_{x\to -2} f(x) = -2$

K) $\lim_{x\to 3} f(x) = -2$

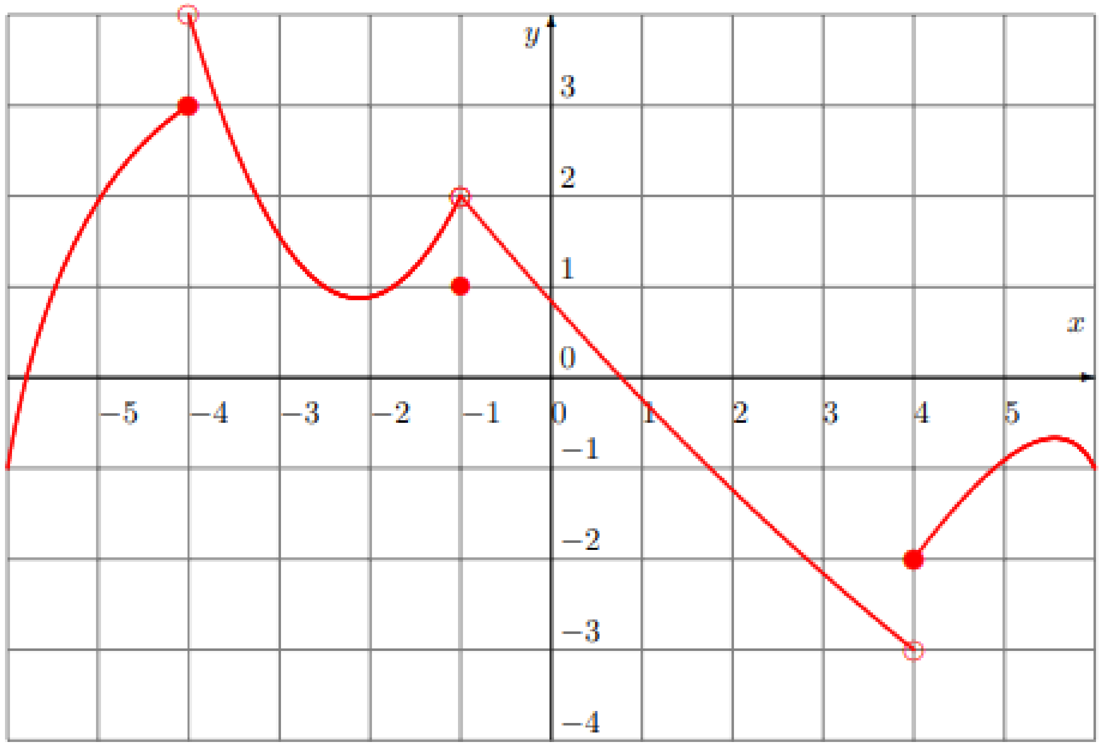
D) $f(-4) = 2$

H) $f(-2) = -2$

L) $f(3) = -3$

Example 1 (From Worksheet)

1. Consider the following function defined by its graph:



Find the following limits:

A) $\lim_{x \to -4^-} f(x) =$ 3

B) $\lim_{x \to -4^+} f(x) =$ 4

C) $\lim_{x \to -4} f(x) =$ DNE

D) $f(-4) =$ 3

E) $\lim_{x \to -1^-} f(x) =$ 2

F) $\lim_{x \to -1^+} f(x) =$ 2

G) $\lim_{x \to -1} f(x) =$ 2

H) $f(-1) =$ 1

I) $\lim_{x \to 4^-} f(x) =$ −3

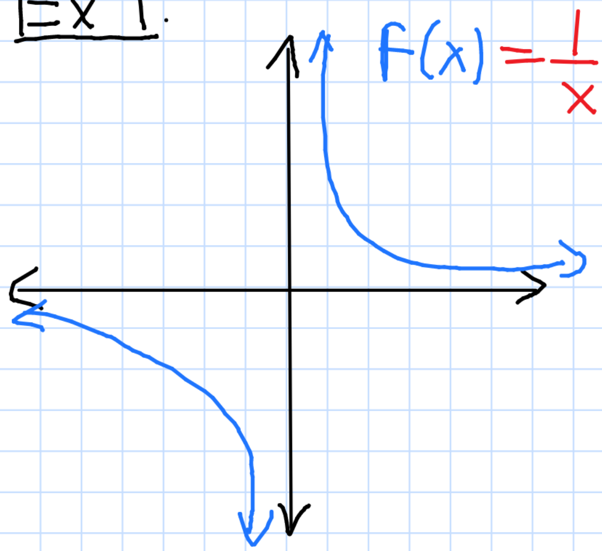
J) $\lim_{x \to 4^+} f(x) =$ −2

K) $\lim_{x \to 4} f(x) =$ DNE

L) $f(4) =$ −2

Lesson 3: Finding Limits Graphically

Ex 1:



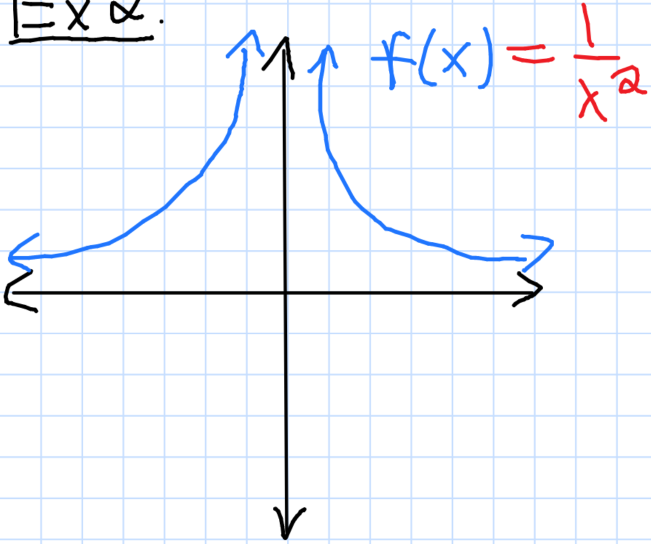
ⓐ $\lim_{x \to 0^-} f(x) =$ $-\infty$

ⓑ $\lim_{x \to 0^+} f(x) =$ ∞

ⓒ $\lim_{x \to 0} f(x) =$ DNE

ⓓ $f(0) =$ undefined

Ex 2:



ⓐ $\lim_{x \to 0^-} f(x) =$ ∞

ⓑ $\lim_{x \to 0^+} f(x) =$ ∞

ⓒ $\lim_{x \to 0} f(x) =$ ∞

ⓓ $f(0) =$ undefined