MA 16010 LESSONS 20+21: OPTIMIZATION

Optimization Problems: Often this includes finding the maximum or minimum value of some function.

- e.g. The minimum time to make a certain journey,
- e.g. The minimum cost for constructing some object,
- e.g. The maximum profit to gain for a business, and so on.

How do we solve an optimization problem?

- Determine a function **(known as objective function)** that we need to maximize or minimize.

- Determine if there are some constraints on the variables. (The equations that describe the constraints are called **the constraint equations**.)
  - If there are constraint equations, rewrite the **objective function** as a function of only one variable.

- Then we can solve for absolute maximum or minimum like we did before.
  - Using either the First Derivative Test or Second Derivative Test.

Recipe for Solving an Optimization Problem

Step 1: Identify what quantity you are trying to optimize.

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Step 6: Reread the question and be sure you have answered exactly what was asked.

Example 1: Of all the numbers whose sum is 50, find the two that have the maximum product. Let x and y be such #s.

Step 1: Identify what quantity you are trying to optimize. Product, P

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

N/A

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$P = xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$x + y = 50$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve (4) for y.

$$y = 50 - x$$

Plug y into (3).

$$P = xy$$
$$= x(50 - x)$$
$$= 50x - x^2$$

Find P' and set $= 0$.

$$P' = 50 - 2x = 0$$
$$50 = 2x$$
$$25 = x$$

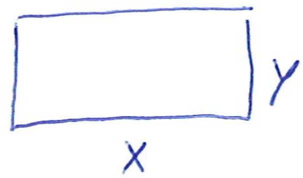
Step 6: Reread the question and be sure you have answered exactly what was asked.

$$x = 25 \Rightarrow y = 50 - x$$
$$= 50 - 25$$
$$y = 25$$

Example 2: A carpenter is building a rectangular room with a fixed perimeter of 54 feet. What are the dimensions of the largest room that can be built? What is its area?

Step 1: Identify what quantity you are trying to optimize. Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$2x + 2y = 54 \iff x + y = 27$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y.

$$y = 27 - x$$

Plug y into ③

$$A = x(27 - x)$$
$$= 27x - x^2$$

Find A' and set $= 0$.

$$A' = 27 - 2x = 0$$
$$x = 27/2$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

$$x = \frac{27}{2} \Rightarrow y = 27 - x, \quad y = \frac{27}{2}$$

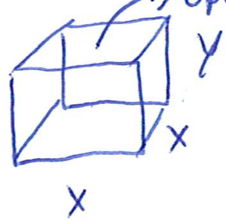
$$\text{Area} = \frac{27}{2} \times \frac{27}{2} = \frac{749}{4}$$

Example 3: An open-top box with a square base is to have a volume of 8 cubic feet. Find the dimensions of the box that can be made with the least amount of material.

Step 1: Identify what quantity you are trying to optimize. Surface Area, A

$\Rightarrow$ open-top i.e. no top

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = x^2 + 4xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$8 = V = x^2 y$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y.

$$\frac{8}{x^2} = y$$

Plug y into ③.

$$A = x^2 + 4x\left(\frac{8}{x^2}\right)$$

$$= x^2 + \frac{32}{x}$$

Rewrite A to be

$$A = x^2 + 32x^{-1}$$

Find A′ and set = 0.

$$A' = 2x - 32x^{-2} = 0$$

$$2x - \frac{32}{x^2} = 0$$

$$\frac{2x}{1} = \frac{32}{x^2}$$

$$2x^3 = 32$$

$$x^3 = 16$$

$$x = \sqrt[3]{16}$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

$$x = \sqrt[3]{16} \Rightarrow y = \frac{8}{x^2}$$

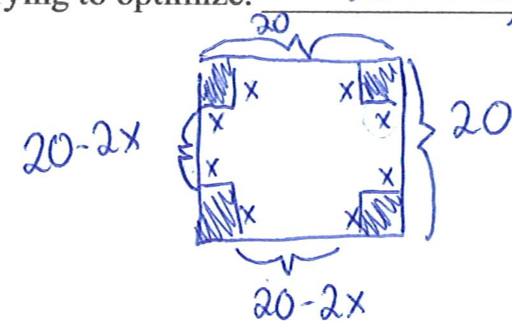
$$= \frac{8}{16^{2/3}}$$

Dimensions: $\sqrt[3]{16} \times \sqrt[3]{16} \times \frac{8}{16^{2/3}}$

Example 4: From a thin piece of cardboard 20 in by 20 in, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

Step 1: Identify what quantity you are trying to optimize. Volume, V

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$V = x(20-2x)^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

None since V is in terms of x only.

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Find V' and set $=0$.

$$V' = 1\cdot(20-2x)^2 + x\cdot 2(20-2x)\cdot(-2) = 0$$

$$(20-2x)\left[20-2x-4x\right] = 0$$

$$(20-2x)(20-6x) = 0$$

$$20-2x=0 \qquad 20-6x=0$$

$$\cancel{x=10} \qquad x = \frac{20}{6} = \frac{10}{3}$$

b/c if $x=10$, we will have no length or width.

Step 6: Reread the question and be sure you have answered exactly what was asked.

$x = \text{height} = \frac{10}{3}$

$$\text{width} = \text{length} = 20-2x = 20-2\left(\tfrac{10}{3}\right) = \frac{40}{3}$$

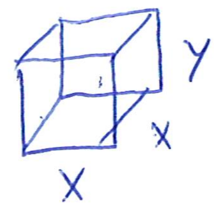
Dimensions: $\frac{10}{3} \times \frac{40}{3} \times \frac{40}{3}$

Volume: $\frac{16000}{27}$

Example 5: A rectangular box has a square base. If the sum of the height and the perimeter of the square base is 20 in, what is the maximum possible volume?

Step 1: Identify what quantity you are trying to optimize. Volume, V

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$V = x^2 y$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$y + 4x = 20$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y.

$$y = 20 - 4x$$

Plug y into ③.

$$V = x^2(20 - 4x)$$
$$= 20x^2 - 4x^3$$

Find V' and set $= 0$.

$$V' = 40x - 12x^2 = 0$$
$$4x(10 - 3x) = 0$$
$$4x = 0 \qquad 10 - 3x = 0$$
$$\cancel{x = 0} \qquad x = \frac{10}{3}$$

b/c if not we have no base.

Step 6: Reread the question and be sure you have answered exactly what was asked.

$$x = \frac{10}{3} \Rightarrow y = 20 - 4x$$
$$= 20 - 4\left(\frac{10}{3}\right)$$
$$= 20/3$$

$$\text{Volume} = x^2 y$$
$$= \left(\frac{10}{3}\right)^2 \cdot \frac{20}{3} = \frac{2000}{27}$$

Example 6: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

b) What price, p, should the company charge to maximize their profit?

Step 1: Identify what quantity you are trying to optimize. Profit, P

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$P = R - C = pq - 5q = (p-5)q$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$q = 2400 - 200p$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Plug 4 into 3,

$$P = (p-5)(2400-200p)$$
$$= -200(p-5)(p-12)$$
$$= -200(p^2 - 17p + 60)$$

Find P' and set $=0$,

$$P' = -200(2p-17) = 0$$
$$p = \frac{17}{2} = 8.5$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

$p = \$8.50$

Example 6: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs $5 to make.

a) What price, p, should the company charge to maximize their revenue?

Step 1: Identify what quantity you are trying to optimize. Revenue, R

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$R = pq$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$q = 2400 - 200p$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Plug q into (3).

$$R = p(2400 - 200p)$$
$$= 2400p - 200p^2$$

Find R' and set $= 0$.

$$R' = 2400 - 400p = 0$$
$$p = 6$$

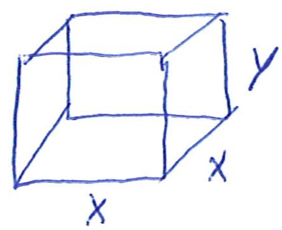
Step 6: Reread the question and be sure you have answered exactly what was asked.

$6

Example 7: A rectangular box is to have a square base and a volume of 800 ft^3. If the material for the base costs \$2 per square foot, the material for the sides costs \$4 per square foot, and the material for the top costs \$1 per square foot, determine the minimum cost for constructing such a box.

Step 1: Identify what quantity you are trying to optimize. Cost, C

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$C = \$2 \cdot x^2 + \$4(4xy) + \$1 \cdot x^2 = 3x^2 + 16xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$800 = x^2 y$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y.

$$y = \frac{800}{x^2}$$

Plug y into ③.

$$C = 3x^2 + 16x\left(\frac{800}{x^2}\right)$$

$$= 3x^2 + \frac{12800}{x}$$

$$= 3x^2 + 12800x^{-1}$$

Find C′ and set = 0.

$$C' = 6x - 12800x^{-2} = 0$$

$$6x - \frac{12800}{x^2} = 0$$

$$6x = \frac{12800}{x^2}$$

$$6x^3 = 12800$$

$$x^3 = \frac{6400}{3} \Rightarrow x = \sqrt[3]{\frac{6400}{3}}$$

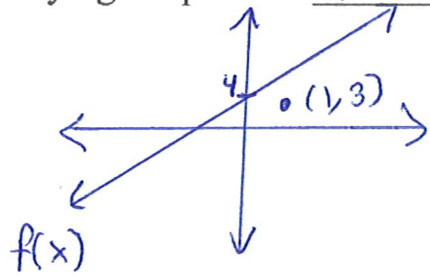
Step 6: Reread the question and be sure you have answered exactly what was asked.

$$x = \sqrt[3]{\frac{6400}{3}} \Rightarrow y = \frac{800}{\left(\frac{6400}{3}\right)^{2/3}} \Bigg\} \quad C = \$62.14$$

Example 8: Find the point on the graph of $f(x) = 2x + 4$ that is the closest to the point $(1,3)$.

Step 1: Identify what quantity you are trying to optimize. Distance, D

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$D = (x-1)^2 + (y-3)^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$y = 2x + 4$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Plug 4 into 3.

$$D = (x-1)^2 + (2x+4-3)^2$$
$$= (x-1)^2 + (2x+1)^2$$

Find D' and set $= 0$.

$$D' = 2(x-1) + 2(2x+1) \cdot 2 = 0$$
$$2x - 2 + 4(2x+1) = 0$$
$$2x - 2 + 8x + 4 = 0$$
$$10x + 2 = 0$$
$$x = -1/5$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

$x = \frac{-1}{5} \Rightarrow y = 2x + 4 = \frac{18}{5}$ $\Big\}$ $\left(-\frac{1}{5}, \frac{18}{5}\right)$