Lesson 22: Antiderivatives and Indefinite Integration Pt 1

Consider the equation $F'(x) = f(x)$. There are two ways to interpret this:

① $f(x)$ is the derivative of $F(x)$

② $F(x)$ is the antiderivative of $f(x)$.

Notation: $F(x) = \int f(x)\,dx$

With antiderivatives start with $f(x)$ and find $F(x)$.

Example 1: ⓐ Differentiate $F(x) = x^2 + 2$

$F'(x) = 2x$

ⓑ Find $\int 2x\,dx$.

What function $F(x)$ has $2x$ as its derivative?

By ⓐ, one such $F(x)$ is $x^2 + 2$.

But so are

- x^2
- $x^2 - 1234$
- $x^2 + (\text{constant})$

Why? The derivative of a constant is zero.

To account for this, use C as an arbitrary constant.

$$\boxed{\int 2x\,dx = x^2 + C}$$

The process of finding all the antiderivatives of a function is called indefinite integration.

Denoted by $\underbrace{\int f(x)\,dx}_{\text{reads as "integral of } f(x)\text{"}} = F(x) + C$ where C is a constant.

- $\int$ integral sign
- $f(x)$ integrand
- x integration variable
- C constant of integration.

Differentiation Rule	Integration Rule
$\frac{d}{dx}(c) = 0$	$\int 0 \, dx = c$
$\frac{d}{dx}(kx) = k$	$\int k \, dx = kx + C$
$\frac{d}{dx}(kf(x)) = kf'(x)$	$\int kf'(x)\, dx = k\int f'(x)\, dx$ $= kf(x) + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int nx^{n-1}\, dx = x^n + C$
$\frac{d}{dx}(x^{n+1}) = (n+1)x^n$	$\int (n+1)x^n \, dx = x^{n+1} + C$
	$(n+1)\int x^n \, dx = x^{n+1} + C$
	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Example 2: Find the indefinite integral

ⓐ $\int (x^2 + 2\sqrt{x})\, dx$

$$\int (x^2 + 2x^{1/2})\, dx = \frac{x^{2+1}}{2+1} + 2\,\frac{x^{1/2+1}}{1/2+1} + C$$

$$= \frac{x^3}{3} + \frac{2\,x^{3/2}}{3/2} + C$$

$$= \frac{x^3}{3} + 2 \cdot \frac{2}{3} x^{3/2} + C$$

$$= \frac{x^3}{3} + \frac{4}{3} x^{3/2} + C$$

ⓑ $\int \left(\frac{1}{\sqrt{x}} + \sqrt[3]{x^2} \right) dx$

$$\int \left(x^{-1/2} + x^{2/3} \right) dx = \frac{x^{-1/2+1}}{-1/2+1} + \frac{x^{2/3+1}}{2/3+1} + C$$

$$= \frac{x^{1/2}}{1/2} + \frac{x^{5/3}}{5/3} + C$$

$$= \frac{2}{1} x^{1/2} + \frac{3}{5} x^{5/3} + C$$

(c) $\int \frac{x^6+x^4}{\sqrt{x}}\,dx$

$$\int \frac{x^6+x^4}{x^{1/2}}\,dx = \int\left(\frac{x^6}{x^{1/2}}+\frac{x^4}{x^{1/2}}\right)dx$$

$$= \int\left(x^{11/2}+x^{7/2}\right)dx$$

$$= \frac{x^{11/2+1}}{11/2+1}+\frac{x^{7/2+1}}{7/2+1}+C$$

$$= \frac{x^{13/2}}{13/2}+\frac{x^{9/2}}{9/2}+C$$

$$= \frac{2}{13}x^{13/2}+\frac{2}{9}x^{9/2}+C$$

Differentiation Rule	Integration Rule
$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}(\sec x) = \sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}(\csc x) = -\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + C$

Example 2: Find the indefinite integral

(a) $\int \frac{\sin x+\cos x}{2}\,dx = \frac{1}{2}\int(\sin x+\cos x)\,dx = \frac{1}{2}\left[\underbrace{\int \sin x\,dx}_{-\cos x} + \underbrace{\int \cos x\,dx}_{\sin x}\right]$

$$= \frac{1}{2}(-\cos x+\sin x)+C$$

$$= -\frac{\cos x}{2}+\frac{\sin x}{2}+C$$

ⓑ $\int \sec x(6\sec x + 8\tan x)\,dx$

$\int (6\sec^2 x + 8\tan x\sec x)\,dx$

$= 6\underbrace{\int \sec^2 x\,dx}_{\tan x} + 8\underbrace{\int \tan x\sec x\,dx}_{\sec x}$

$= 6\tan x + 8\sec x + C$

ⓒ $\int (8\cot x\sin x + 5)\,dx$

$\int \left(8\,\frac{\cos x}{\sin x}\cdot \sin x + 5\right)dx = \int (8\cos x + 5)\,dx$

$= 8\sin x + 5x + C$

Differentiation Rule	Integration Rule
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}(\ln x) = \frac{1}{x},\ x>0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Example 3: Find the indefinite integral

ⓐ $\int \frac{5}{x}\,dx$

$5\int \frac{1}{x}\,dx = 5\ln\lvert x\rvert + C$

ⓑ $\int \frac{1+4xe^x}{x}\,dx$

$\int \left(\frac{1}{x} + \frac{4xe^x}{x}\right)dx = \int \frac{1}{x}\,dx + \int 4e^x\,dx$

$= \ln\lvert x\rvert + 4e^x + C$

ⓒ $\int \left(x^e + \frac{1}{x} + 1\right)dx$

$\int x^e\,dx + \int \frac{1}{x}\,dx + \int 1\,dx = \frac{x^{e+1}}{e+1} + \ln\lvert x\rvert + x + C$