Lesson 23: Antiderivatives and Indefinite Integration Pt 2

A differential equation in x and y is an equation that relates x, y, and y'.

Example 1: Solve the differential equation $y' = 3x$.

$$\int y' \, dx = \int 3x \, dx$$

$$\int \frac{dy}{\cancel{dx}} \cancel{dx} = \int 3x \, dx$$

$$\int dy = \int 3x \, dx$$

$$y = \frac{3x^2}{2} + C$$

Recall $y' = \dfrac{dy}{dx}$

This solution is also called the general solution.

But what if I give you a y-value at some x?
We call that the initial condition.
The answer of a specific function is particular solution.
A differential equation with initial condition is an Initial Value Problem (IVP).

Example 2: Solve IVP, $y' = 3x$ with $y(0) = 2$.

By Ex 1, $y = \dfrac{3x^2}{2} + C$

So with $y(0) = 2$,

$$2 = y(0) = \frac{3(0)^2}{2} + C$$

$$2 = 0 + C$$

$$2 = C$$

Plug $C = 2$ into $y = \dfrac{3x^2}{2} + C$

$$y = \frac{3x^2}{2} + 2$$

Example 3: Solve the IVP: $y' = \frac{9}{x}$ and $y(e) = 52$

Integrate each side: $\int y' \, dx = \int \frac{9}{x} \, dx$

$$y = 9\ln|x| + C$$

Plug the condition $y(e) = 52$ and find C.

$$52 = y(e) = 9\ln|e| + C$$
$$52 = 9 + C$$
$$43 = C$$

Plug $C = 43$ into y: $y = 9\ln|x| + 43$

Example 4: Solve the IVP: $y' = 7\cos(x) + 3$ and $y(\pi) = 3$

Integrate each side: $\int y' \, dy = \int (7\cos(x) + 3) \, dx$

$$y = 7\sin x + 3x + C$$

Plug the condition $y(\pi) = 3$ and find C.

$$3 = y(\pi) = 7\sin(\pi) + 3\pi + C$$
$$3 = 0 + 3\pi + C$$
$$3 - 3\pi = C$$

Plug $C = 3 - 3\pi$ into y: $y = 7\sin x + 3x + 3 - 3\pi$

Example 5: Solve the IVP: $2y'' = e^x + 4$, $y'(0) = 5$, $y(2) = 10$

Integrate each side: $\int 2y'' \, dy = \int (e^x + 4) \, dx$

$$2y' = e^x + 4x + C$$

Plug the condition $y'(0) = 5$ and find C.

$$2(5) = e^0 + 4 \cdot (0) + C$$
$$10 = 1 + C$$
$$9 = C$$

Plug $C = 9$ into y': $2y' = e^x + 4x + 9$

Now repeat.

Integrate each side: $\int 2y'\,dy = \int (e^x + 4x + 9)\,dx$

$$2y = e^x + \frac{4x^2}{2} + 9x + C$$

$$2y = e^x + 2x^2 + 9x + C$$

Plug the condition $y(2) = 10$ and find C.

$$2(10) = e^2 + 2(2)^2 + 9(2) + C$$

$$20 = e^2 + 8 + 18 + C$$

$$-e^2 - 6 = C$$

Plug $C = -e^2 - 6$ into y:

$$2y = e^x + 2x^2 + 9x - e^2 - 6$$

$$y = \frac{e^x}{2} + \frac{2x^2}{2} + \frac{9x}{2} - \frac{e^2}{2} - \frac{6}{2}$$

$$y = \frac{e^x}{2} + x^2 + \frac{9x}{2} - \frac{e^2}{2} - 3$$

Example 6: The rate of growth $\frac{dP}{dt}$ of a population of bacteria is proportional to the square root of t with a constant coefficient of 5, where P is the population size and t is the time in days $(0 \le t \le 10)$. The initial size of the population is 200. Approximate the population after 7 days, i.e. $\frac{dP}{dt} = 5\sqrt{t}$ and $P(0) = 200$

Integrate each side: $\int \frac{dP}{dt}\,dt = \int 5t^{1/2}\,dt$

$$\int dP = \int 5t^{1/2}\,dt$$

$$P = 5\frac{t^{1/2+1}}{1/2+1} + C$$

$$P = 5 \cdot \frac{t^{3/2}}{3/2} + C$$

$$P = 5 \cdot \frac{2}{3} t^{3/2} + C$$

$$P = \frac{10}{3} t^{3/2} + C$$

Plug the condition
$P(0)=200$ and find C.

$$200 = \frac{10}{3}(0)^{3/2} + C$$

$$200 = C$$

Plug $C=200$ into P.

$$P = \frac{10}{3}t^{3/2} + 200$$