Lesson 24: Area and Riemann Sums

Let's recap some basics about sums.

Example 1: Evaluate $\sum_{i=1}^{5} i^2$

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

Example 2: Use the sigma notation to write the sum of

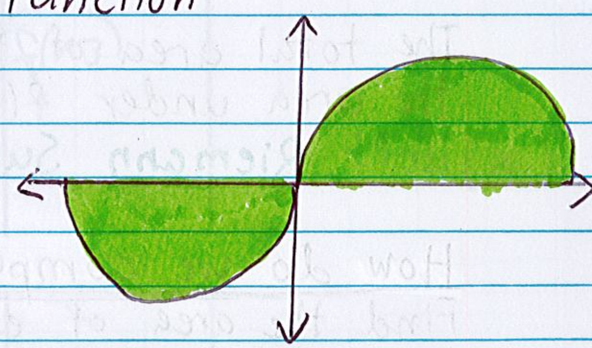
ⓐ $2(1^3+1) + 2(2^3+1) + \dots + 2(n^3+1) = \sum_{i=1}^{n} 2(i^3+1)$

ⓑ $\frac{2}{0+5} + \frac{2}{1+5} + \dots + \frac{2}{n+5} = \sum_{i=0}^{n} \frac{2}{i+5}$

ⓒ $(3\sqrt{0}+1) + (3\sqrt{1}+2) + \dots + (3\sqrt{n}+n+1) = \sum_{i=0}^{n} (3\sqrt{i}+i+1)$

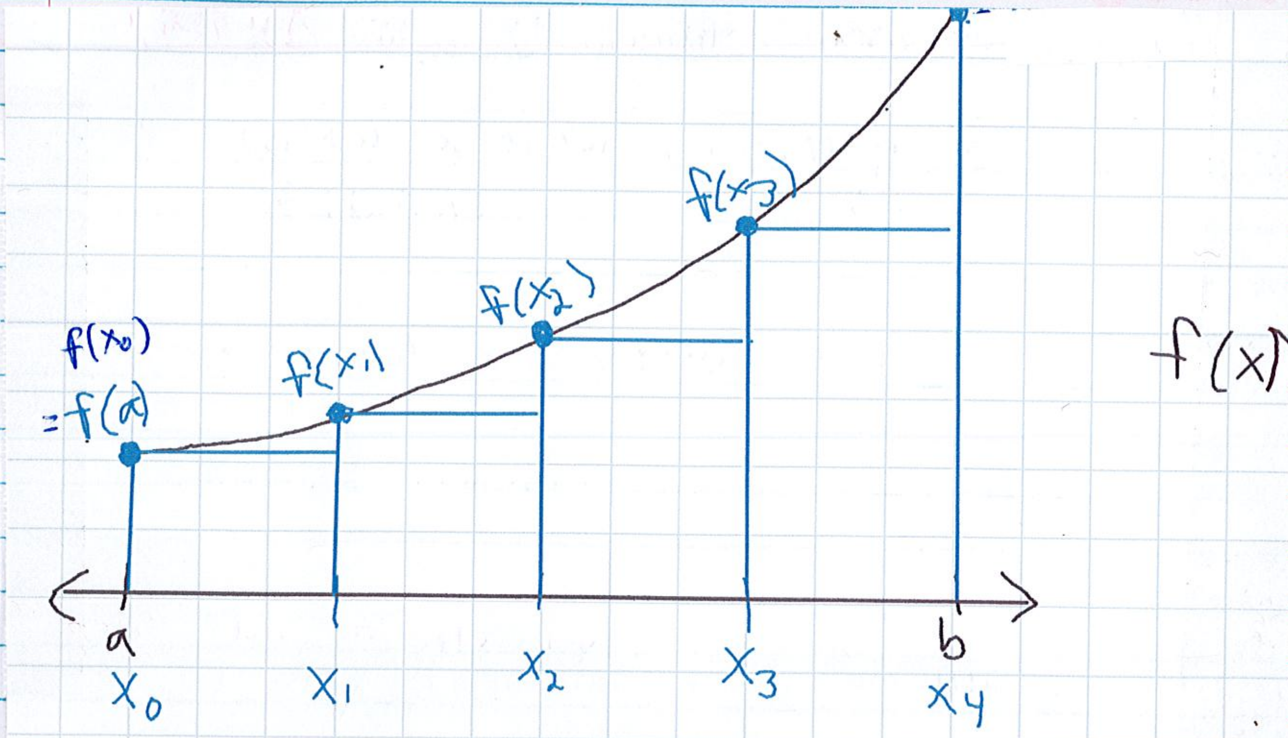
So why do we care about sums and sigmas? It's because we can use them to estimate the Signed Area under the curve of a function

Signed Area is the area enclosed by the function and the x-axis with a sign.



- ↳ If the function is above the x-axis, then the area is positive
- ↳ If the function is below the x-axis, then the area is negative.

We want to estimate the area under the f(x), given in the next page, from a to b by using 4 rectangles.



We created this image by

- Dividing the interval $[a,b]$ into 4 sub-intervals
- Then construct the rectangles by starting at the points on $f(x)$ whose x coordinates are the left end of each of the sub-intervals.

The total area of these rectangles gives us an estimate of the area under $f(x)$. A sum like this is called the **Left Riemann Sum.**

How do we compute this sum?

Find the area of each rectangle and sum them up. But let's work smart not hard.

Width of each rectangle is the same

$$w = \Delta x = \frac{b-a}{4}$$

Length of 1^{st} rectangle is $f(a) = f(x_0)$

Length of 2^{nd} rectangle is $f(x_1)$

$\vdots$ $\qquad\qquad$ $\vdots$

Length of i^{th} rectangle is $f(x_i)$

So the area of each rectangle is $lw = \Delta x\, f(x_i)$

4 rectangles $\Rightarrow \sum_{i=0}^{3} f(x_i)\Delta x$ where $x_i = a + i\Delta x$

$\Delta x = \frac{b-a}{n}$

In General, the Left Riemann Sum for n rectangles is

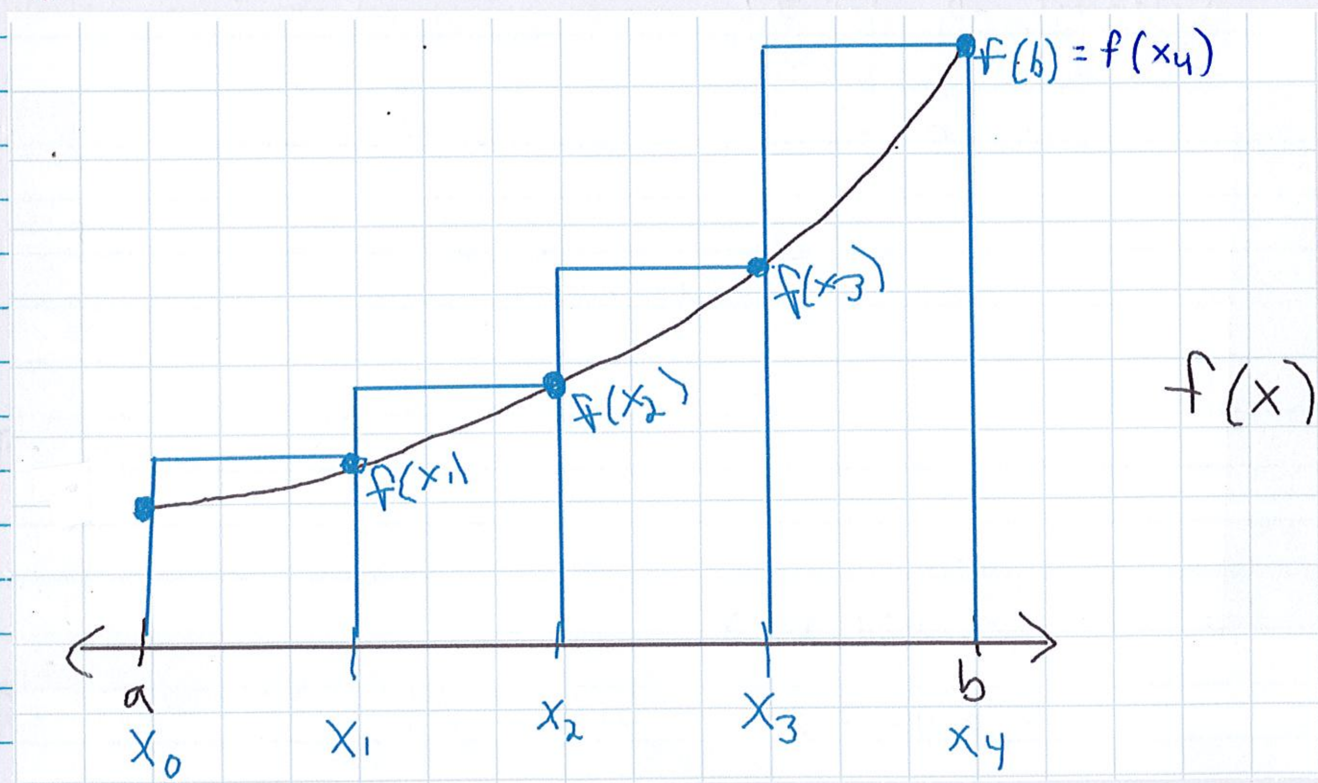
$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$ where $x_i = a + i\Delta x$

$\Delta x = \frac{b-a}{n}$

Can we do this by using right end of each sub-interval?

Yes. It's called Right Riemann Sum.

Below you can see, the image corresponding with a Right Riemann Sum.



Remember these rectangles are created from right to left!!!

Notice that the width is the same, the lengths are $f(x_1)$, $f(x_2)$, $f(x_3)$, and $f(x_4)$. So the Right Riemann Sum for this image is

$$\sum_{i=1}^{4} f(x_i)\Delta x \quad \text{where } x_i = a + i\Delta x$$
$$\Delta x = \frac{b-a}{n}$$

In General, the Right Riemann Sum for n rectangles is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where } x_i = a + i\Delta x$$
$$\Delta x = \frac{b-a}{n}$$

Note the only difference both formula is the index.

Example 3: Use the Left Riemann Sum with 3 rectangles to estimate the area under the curve $y = x^2$ on the interval of $[1, 7]$.

$$a = 1 \qquad b = 7 \qquad n = 3$$
$$\Delta x = \frac{b-a}{n} = \frac{7-1}{3} = \frac{6}{3} = 2$$
$$x_i = a + i\Delta x = 1 + 2i$$
$$f(x_i) = f(1+2i) = (1+2i)^2$$

Recall $L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$

$$L_3 = \sum_{i=0}^{3-1} 2(1+2i)^2$$
$$= \sum_{i=0}^{2} 2(1+2i)^2$$
$$= 2(1+2(0))^2 + 2(1+2(1))^2 + 2(1+2(2))^2 = 70$$

Example 4: Use the Right Riemann Sum with 4 rectangles to estimate the area under the curve $y = \sqrt{x} - x$ on $[0, 4]$

$a = 0 \quad b = 4 \quad n = 4$

$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

$x_i = a + i\Delta x = 0 + i(1) = i$

$f(x_i) = f(i) = \sqrt{i} - i$

Recall $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$R_4 = \sum_{i=1}^{4} [\sqrt{i} - i] \cdot 1$

$= (\sqrt{1} - 1) + (\sqrt{2} - 2) + (\sqrt{3} - 3) + (\sqrt{4} - 4) = \sqrt{2} + \sqrt{3} - 7$

Example 5: Use the Left/Right Riemann Sum with 50 rectangles to estimate the area under the curve $y = x^2 - 2x + 1$ on $[0, 10]$ Write your answer using sigma notation.

$a = 0 \quad b = 10 \quad n = 50$

$\Delta x = \frac{b-a}{n} = \frac{10-0}{50} = \frac{1}{5}$

$x_i = a + i\Delta x = 0 + i(1/5) = i/5$

$f(x_i) = f\left(\frac{i}{5}\right) = \left(\frac{i}{5}\right)^2 - 2\left(\frac{i}{5}\right) + 1 = \frac{i^2}{25} - \frac{2i}{5} + 1$

Recall $L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$

$L_{50} = \sum_{i=0}^{50-1} \left(\frac{i^2}{25} - \frac{2i}{5} + 1\right) \cdot \frac{1}{5} = \sum_{i=0}^{49} \frac{1}{5}\left(\frac{i^2}{25} - \frac{2i}{5} + 1\right)$

Recall $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$R_{50} = \sum_{i=1}^{50} \frac{1}{5}\left(\frac{i^2}{25} - \frac{2i}{5} + 1\right)$

Example 6: Use the Left/Right Riemann Sum with 80 rectangles to estimate the area under the curve $y = e^{3x} - 18$ on $[10, 20]$. Write your answer using sigma notation.

$a = 10 \quad b = 20 \quad n = 80$

$$\Delta x = \frac{b-a}{n} = \frac{20-10}{80} = \frac{10}{80} = \frac{1}{8}$$

$$x_i = a + i\Delta x = 10 + i(1/8) = 10 + \frac{i}{8}$$

$$f(x_i) = f\left(10 + \frac{i}{8}\right) = \exp\left[3\left(10 + \frac{i}{8}\right)\right] - 18$$

Recall $L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$

$$L_{80} = \sum_{i=0}^{79} \left(\exp\left[3\left(10 + \frac{i}{8}\right)\right] - 18\right) \cdot \frac{1}{8}$$

Recall $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$$R_{80} = \sum_{i=1}^{80} \left(\exp\left[3\left(10 + \frac{i}{8}\right)\right] - 18\right) \cdot \frac{1}{8}$$