Lesson 25: Definite Integrals

When the # of rectangles used gets bigger and bigger, the approximation gets better and better.

i.e. the approximation gets closer and closer to the exact signed area.

So when $n \to \infty$, the Left/Right Riemann Sum approaches the actual signed area.

$$\text{Signed Area} = \int_a^b f(x)\,dx$$

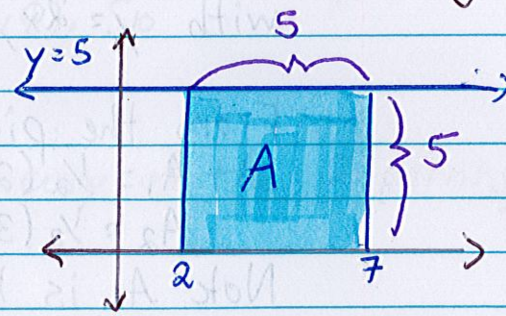
a - lower limit of integration
b - upper limit of integration

Definition: An integral with lower and upper limits is called a definite integral.

So no more $+C$, when you see lower/upper limits.

Example 1: Using Geometric Formulas, evaluate the following definite integrals.

ⓐ $\int_2^7 5\,dx$

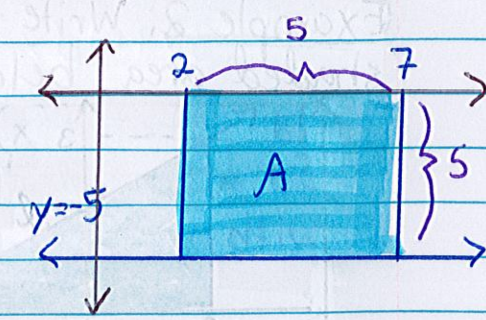


Note that $f(x)=5$ for $2 \le x \le 7$.
So we need to draw a picture with $y=5$, $x=2$, and $x=7$.

From the picture, $A = 5 \times 5 = 25$. So

$$\int_2^7 5\,dx = 25$$

ⓑ $\int_2^7 -5\,dx$

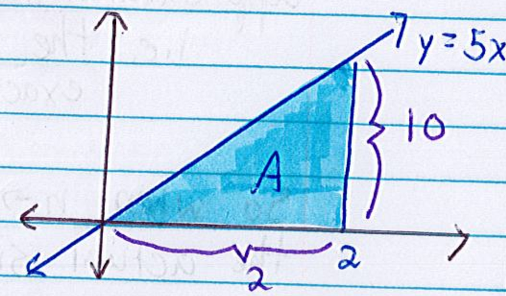


Note that $f(x) = -5$ for $2 \le x \le 7$.
So we need to draw a picture with $y=5$, $x=2$, and $x=7$.

From the picture, $A = 5 \times 5 = 25$, but it's beneath the x-axis. So it gets a negative.

Hence $\int_2^7 -5\,dx = -25$

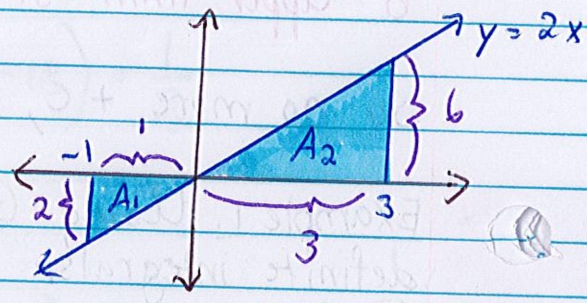
ⓒ $\int_0^2 5x\,dx$



Note that $f(x)=5x$ for $0 \le x \le 2$.
So we need to draw a picture with $y=5x$, $x=0$, and $x=2$

From the picture, $A = \frac{1}{2}(2)(10) = 10$, So
$$\int_0^2 5x\,dx = 10$$

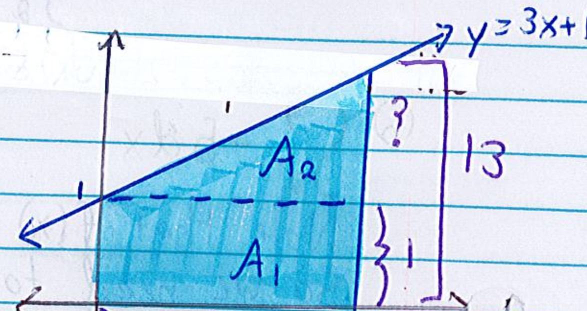
ⓓ $\int_{-1}^3 2x\,dx$



Note that $f(x)=2x$ for $-1 \le x \le 3$.
So we need to draw a picture with $y=2x$, $x=-1$, and $x=3$.

From the picture, we have 2 triangles.
$A_1 = \frac{1}{2}(2)(1) = 1$
$A_2 = \frac{1}{2}(3)(6) = 9$
Note A_1 is below the x-axis so it's -1
A_2 is above the x-axis so it's $+9$
So $\int_{-1}^3 2x\,dx = -1+9 = 8$

ⓔ $\int_0^4 (3x+1)\,dx$



Note that $f(x)=3x+1$ for $0 \le x \le 4$
So we need to draw a picture with $y=3x+1$, $x=0$, and $x=4$

From the picture, we have a rectangle and a triangle.
$A_1 = 1(4) = 4$

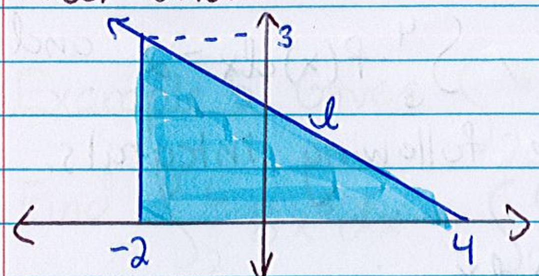
For the height of A_2 we need to solve an equation $?+1=13$
So the height of A_2 is $?=12$. So

$$A_2 = \tfrac{1}{2}(12)(4) = 24$$

Since both A_1 and A_2 are above the x-axis, just add them.

$$\int_0^4 (3x+1)\,dx = 4+24 = 28$$

Example 2: Write the definite integral that represents the shaded area below.



Coming up with the bounds, is the easiest part. Just look at the x-values. So

$$A = \int_{-2}^{4} l\,dx$$

Next we need to determine the equation of l. Note we have the points $(-2,3)$ and $(4,0)$. So

$$m = \frac{0-3}{4-(-2)} = \frac{-3}{6} = -\frac{1}{2}$$

So $y = \frac{-1}{2}x + b$. To find b use any of the 2 points, and solve for it. With $(4,0)$,

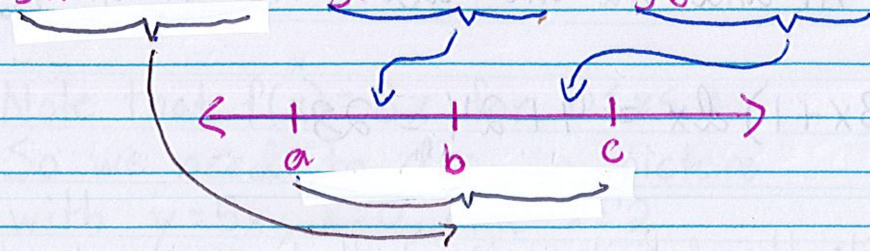
$$0 = -\frac{1}{2}(4) + b$$
$$0 = -2 + b$$
$$b = 2$$

So $y = \frac{-1}{2}x + 2 \Rightarrow A = \int_{-2}^{4}\left(\frac{-1}{2}x + 2\right)dx$

Properties of Definite Integrals

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$



Example 3: Given $\int_1^3 f(x)\,dx = 5$, $\int_3^4 f(x)\,dx = 2$ and $\int_1^3 g(x)\,dx = 10$. Evaluate the following integrals.

ⓐ $\int_1^3 2f(x)\,dx = 2\underbrace{\int_1^3 f(x)\,dx}_{5}$

$= 2(5) = 10$

ⓑ $\int_4^3 f(x)\,dx = -\underbrace{\int_3^4 f(x)\,dx}_{2}$

$= -2$

ⓒ $\int_1^3 [2f(x) - 3g(x)]\,dx = 2\underbrace{\int_1^3 f(x)\,dx}_{5} - 3\underbrace{\int_1^3 g(x)\,dx}_{10}$

$= 2(5) - 3(10)$

$= 10 - 30 = -20$

ⓓ $\int_1^4 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx$

$= 5 + 2 = 7$

Example 4: Given $\int_3^7 x^2\,dx = \frac{316}{3}$, $\int_3^7 x\,dx = 20$, $\int_3^7 dx = 4$

Evaluate $\int_3^7 [-4x^2 + x - 8]\,dx$

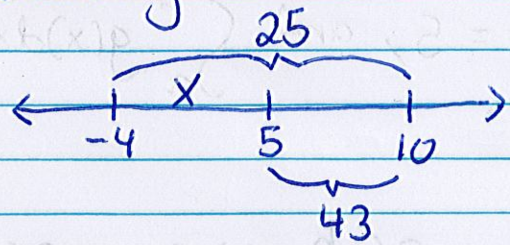
$$\int_3^7 [-4x^2+x-8]\,dx = -4\int_3^7 x^2\,dx + \int_3^7 x\,dx - 8\int_3^7 dx$$
$$= -4\left(\frac{316}{3}\right) + 20 - 8(4)$$
$$= -\frac{1300}{3}$$

Example 5: Given $\int_2^6 2x^3\,dx = 640$.

Find $\int_2^6 8x^3\,dx = \int_2^6 4 \cdot 2x^3\,dx$
$$= 4\int_2^6 2x^3\,dx$$
$$= 4(640)$$
$$= 2560$$

Example 6: Given $\int_{-4}^{10} h(t)\,dt = 25$, $\int_5^{10} h(t)\,dt = 43$. Find $\int_{-4}^{5} h(t)\,dt$.

Drawing a # line will be prudent in such a question.



Let x be $\int_{-4}^{5} h(t)\,dt$.

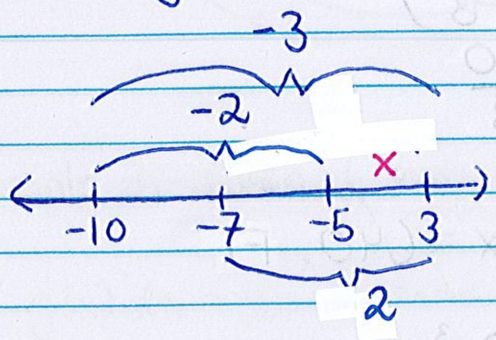
So with the # line we get
$$x + 43 = 25$$
$$x = -18$$

Hence $\int_{-4}^{5} h(t)\,dt = -18$

Example 7: Given $\int_{-7}^{3} f(x)\,dx = 2$, $\int_{-10}^{-5} f(x)\,dx = -2$, $\int_{-10}^{3} f(x)\,dx = -3$. Find the following integrals

ⓐ $\int_{-5}^{3} f(x)\,dx$

Drawing a # line will be prudent in such a question.



Let x be $\int_{-5}^{3} f(x)\,dx$

So with the # line we get

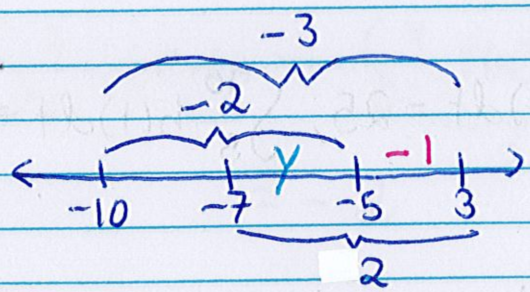
$$-3 = -2 + x$$

$$-1 = x$$

Hence $\int_{-5}^{3} f(x)\,dx = -1$

ⓑ $\int_{-7}^{-5} f(x)\,dx$

Using the # line from ⓐ, Let y be $\int_{-7}^{-5} f(x)\,dx$



So with the # line we get

$$2 = y + (-1)$$

$$3 = y$$

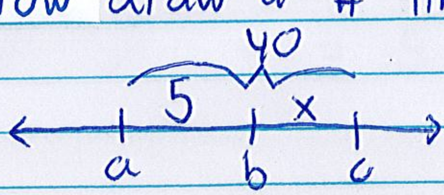
Hence $\int_{-7}^{-5} f(x)\,dx = 3$

Example 8: Given $\int_{a}^{b} g(x)\,dx = 5$, and $\int_{a}^{c} g(x)\,dx = 8\int_{a}^{b} g(x)\,dx$

Compute $\int_{b}^{c} g(x)\,dx$.

First note that $\int_{a}^{c} g(x)\,dx = 8\int_{a}^{b} g(x)\,dx = 8(5) = 40$

Now draw a # line with all your information.



Let x be $\int_{b}^{c} g(x)\,dx$

So with the # line we get

$$5 + x = 40$$

$x = 35 \Rightarrow \int_b^c g(x)\,dx = 35$