Lesson 26: The Fundamental Theorem of Calculus

Recall ① If we want to find all the antiderivatives of $f(x)$, we evaluate $\int f(x)\,dx$

② If we want to find the signed area under a curve of $f(x)$ from a to b, we evaluate $\int_a^b f(x)\,dx$

Note both have an integral signs. But can we combine ① and ②? Yes via The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus (FTC)

Suppose $f(x)$ is continuous on the interval $[a,b]$. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

In pratice, to integrate $\int_a^b f(x)\,dx$, we write

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

We read $\Big]_a^b$ as "$F(x)$ evaluated from a to b"

Example 1: Evaluate

ⓐ $$\int_0^3 2x\,dx = \frac{2x^2}{2}\Big]_0^3$$

$$= x^2\Big]_0^3$$

$$= 3^2 - 0^2$$

$$= 9$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1}$

ⓑ $$\int_4^9 \frac{2x^3+\sqrt{x}}{x^2}\,dx = \int_4^9 \left(\frac{2x^3}{x^2} + \frac{x^{1/2}}{x^2}\right)dx$$

$$= \int_4^9 \left(2x + x^{-3/2}\right)dx$$

$$= \left(\frac{2x^2}{2} + \frac{x^{-1/2}}{-1/2}\right)\Big]_4^9$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= \left(x^2 - \frac{2}{\sqrt{x}} \right) \Big]_4^9$$

$$= \left(9^2 - \frac{2}{\sqrt{9}} \right) - \left(4^2 - \frac{2}{\sqrt{4}} \right)$$

$$= 81 - \frac{2}{3} - (16 - 1)$$

$$= 81 - \frac{2}{3} - 15 = \frac{196}{3}$$

ⓒ $\int_0^{\pi/4} \sec^2 x \, dx = \tan x \Big]^{\pi/4}$ Recall $\int \sec^2 x \, dx = \tan x$

$$= \tan(\pi/4) - \tan(0)$$

$$= 1 - 0$$

$$= 1$$

ⓓ $\int_0^{\pi/2} (\cos x \tan x + 2) \, dx = \int_0^{\pi/2} \left(\cos x \cdot \frac{\sin x}{\cos x} + 2 \right) dx$

$$= \int_0^{\pi/2} (\sin x + 2) \, dx$$

Recall $\int \sin x \, dx = \cos x$

$$= (-\cos x + 2x) \Big]_0^{\pi/2}$$

$$= \left(-\cos\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) \right) - \left(-\cos(0) + 2(0) \right)$$

$$= 0 + \pi - (-1 + 0)$$

$$= \pi + 1$$

ⓔ $\int_0^3 (7e^x + 7) \, dx = (7e^x + 7x) \Big]_0^3$ Recall $\int e^x dx = e^x$

$$= (7e^3 + 7(3)) - (7e^0 + 7(0))$$

$$= 7e^3 + 21 - (7 + 0)$$

$$= 7e^3 + 14$$

Example 2: Find the area of the region bounded by the graph of the following equations:

$$y = 7\left(\frac{\sqrt{x}}{4} - \frac{x}{5}\right)^2, \quad y = 0, \quad x = 1, \quad x = 4$$

Note that $y = 7\left(\frac{\sqrt{x}}{4} - \frac{x}{5}\right)^2$ is above the x-axis, so we have the following integral

$$\int_1^4 7\left(\frac{\sqrt{x}}{4} - \frac{x}{5}\right)^2 dx = \int_1^4 7\left(\left(\frac{\sqrt{x}}{4}\right)^2 - 2\left(\frac{\sqrt{x}}{4}\right)\left(\frac{x}{5}\right) + \left(\frac{x}{5}\right)^2\right) dx$$

$$= \int_1^4 7\left(\frac{x}{16} - \frac{x^{3/2}}{10} + \frac{x^2}{25}\right) dx$$

$$= \int_1^4 \left(\frac{7}{16}x - \frac{7}{10}x^{3/2} + \frac{7}{25}x^2\right) dx$$

$$= \left(\frac{7}{16} \cdot \frac{x^2}{2} - \frac{7}{10} \cdot \frac{x^{5/2}}{5/2} + \frac{7}{25} \cdot \frac{x^3}{3}\right)\Bigg]_1^4$$

$$= \left(\frac{7}{32}x^2 - \frac{7}{10} \cdot \frac{2}{5}x^{5/2} + \frac{7}{75}x^3\right)\Bigg]_1^4$$

$$= \left(\frac{7}{32}x^2 - \frac{14}{50}x^{5/2} + \frac{7}{75}x^3\right)\Bigg]_1^4$$

$$= \left(\frac{7}{32}(4)^2 - \frac{14}{50}(4)^{5/2} + \frac{7}{75}(4)^3\right) - \left(\frac{7}{32} - \frac{14}{50} + \frac{7}{75}\right)$$

$$= \frac{77}{160}$$

MA 16010 LESSON 26: THE FUNDAMENTAL THEOREM OF CALCULUS (PROBLEM SET)

Example 3: **The growth rate of the population of a city is**

$$P'(t) = -500(3 - t)$$

where t is time in years. How does the population change $t = 1$ year to $t = 3$ years?

$$\int_1^3 p'(t)\,dt = \int_1^3 -500(3-t)\,dt$$

$$= -500\int_1^3 (3-t)\,dt$$

$$= -500\left(3t - \frac{t^2}{2}\right)\Big]_1^3$$

$$= -500\left[\left(3(3) - \frac{3^2}{2}\right) - \left(3(1) - \frac{1^2}{2}\right)\right]$$

$$= -500\left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right]$$

$$= -500\left(6 - \frac{8}{2}\right)$$

$$= -500(6-4)$$

$$= -1000$$

Recall

- Displacement is the difference in position
- It could be positive or negative
- The sign indicates the direction

By FTC,

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(t)\Big]_a^b = s(b) - s(a)$$

Example 4: **The velocity function, in feet per second is given for a particle moving along a straight line**

$$v(t) = -10t + 20$$

where t is in seconds.

a) Find the displacement from $t = 0$ to $t = 2$ seconds.

$$\int_0^2 v(t)\,dt = \int_0^2 (-10t + 20)\,dt$$

$$= \left(-\frac{10t^2}{2} + 20t\right)\Big]_0^2$$

$$= (-5t^2 + 20t)\Big]_0^2$$

$$= (-5(2)^2 + 20(2)) - (-5(0)^2 + 20(0))$$

$$= -20 + 40$$

$$= 20$$

b) Find the time t when the displacement is zero after the particle starts moving.

i.e. $\int_0^t v(t)\,dt = 0$

$$\int_0^t (-10t + 20)\,dt = 0$$

From ⓐ

$$(-5t^2 + 20t)\Big]_0^t = 0$$

$$(-5t^2 + 20t) - (\underbrace{\cancel{-5(0)^2 + 20(0)}}_{0}) = 0$$

$$-5t^2 + 20t = 0$$

$$-5t(t-4) = 0$$

$-5t = 0$	$t - 4 = 0$
$t = 0$	$t = 4$

Note we want after the particle is moving. So $t = 4$

Example 5: A faucet is turned on at 9:00 am and water starts to flow into a tank at the rate of

$$r(t) = 6\sqrt{t}$$

where t is time in hours after 9:00 am and the rate $r(t)$ in in cubic feet per hour.

a) How much water, in cubic feet, flows into the rank from 10:00 am to 1:00 pm?

9:00 am $\Rightarrow t=0$
10:00 am $\Rightarrow t=1$
1:00 pm $\Rightarrow t=4$

$$\int_1^4 r(t)\,dt = \int_1^4 6t^{1/2}\,dt$$
$$= \frac{6t^{3/2}}{3/2}\Big]_1^4$$
$$= \frac{2}{3}\cdot 6t^{3/2}\Big]_1^4$$
$$= 4t^{3/2}\Big]_1^4$$
$$= 4(4)^{3/2} - 4(1)^{3/2}$$
$$= 28$$

b) How many hours after 9:00 am will there be 121 cubic feet of water in the tank?

i.e. $\int_0^x r(t)\,dt = 121$

By our integration in ⓐ,

$$4t^{3/2}\Big]_0^x = 121$$
$$4x^{3/2} - 4(0)^{3/2} = 121$$
$$4x^{3/2} = 121$$
$$x^{3/2} = \frac{121}{4}$$
$$\left(x^{3/2}\right)^{2/3} = \left(121/4\right)^{2/3}$$
$$x = \left(121/4\right)^{2/3} \approx 9.7 \text{ hrs}$$