Lesson 27: Numerical Integration

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With the FTC, we are able to evaluate definite integrals for certain integrands.

However, there are many functions that we do not know how to integrate

ex. ① $f(x) = e^x\sqrt{x^2+1}$

② $f(x) = \dfrac{\sin x}{x+1}$

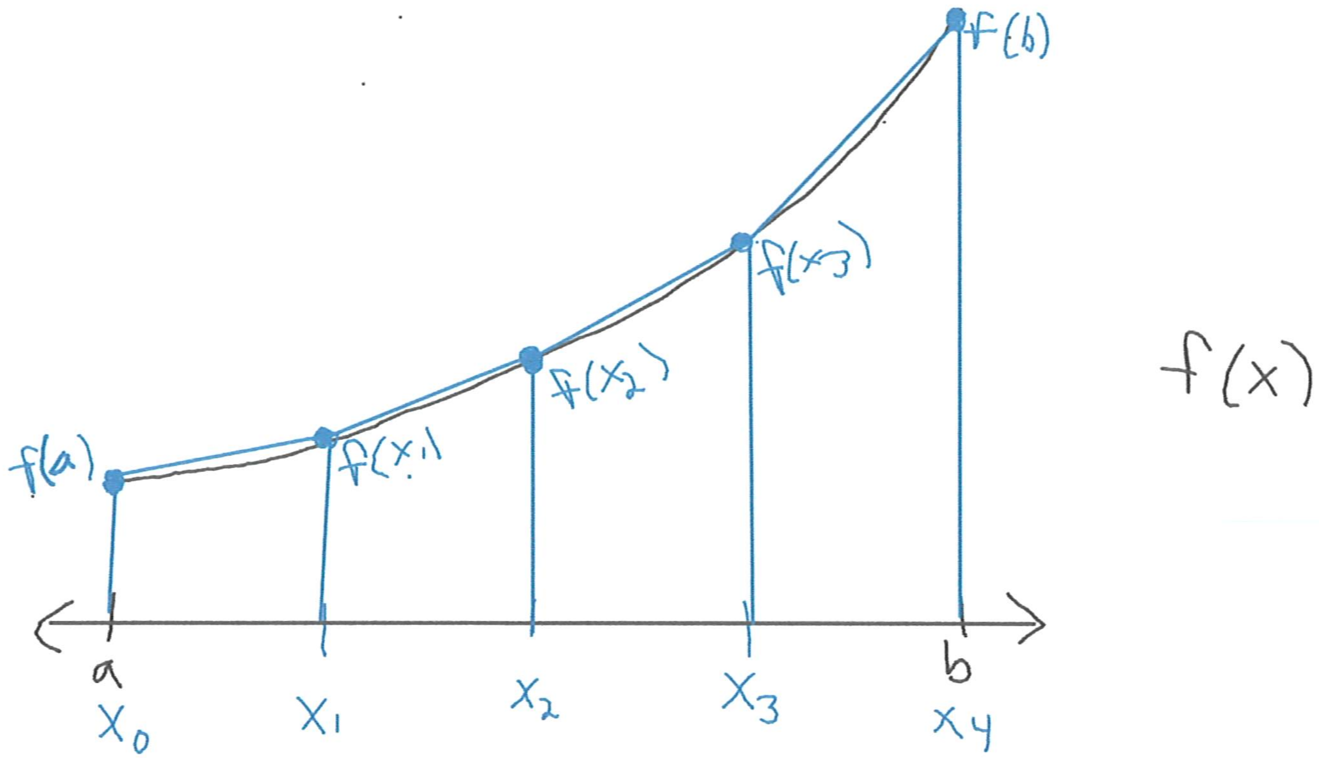
The Trapezoid Rule is an approx method that allows us to approx definite integrals.

Trapezoid Rule is similar to Riemann Sums

Instead of using rectangles, we are using trapezoids.

Suppose $f(x)$ is continuous on $[a,b]$. We want to approx the area

$\int_a^b f(x)dx$ using 4 trapezoids



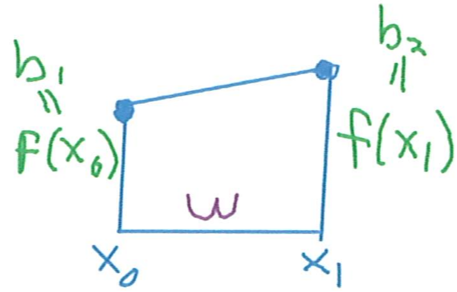
Recall the area of a trapezoid is

$$A = \frac{1}{2}(b_1 + b_2)w$$

Good news: The width of each sub-intervals is the same as in Riemann sums

$$\Delta x = \frac{b-a}{n} = w$$

Let's determine the bases of the first trapezoid



$b_1 = f(x_0)$ and $b_2 = f(x_1)$

Area of 1st trapezoid, t_1, is

$$t_1 = \frac{1}{2}\left(f(x_0) + f(x_1)\right)\Delta x$$

Similarly,

$$t_2 = \frac{1}{2}(f(x_1) + f(x_2))\Delta x$$

$$t_3 = \frac{1}{2}(f(x_2) + f(x_3))\Delta x$$

$$t_4 = \frac{1}{2}(f(x_3) + f(x_4))\Delta x$$

Let's sum all four of these.

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= \frac{1}{2}(f(x_0) + f(x_1))\Delta x \\ &\quad + \frac{1}{2}(f(x_1) + f(x_2))\Delta x \\ &\quad + \frac{1}{2}(f(x_2) + f(x_3))\Delta x \\ &\quad + \frac{1}{2}(f(x_3) + f(x_4))\Delta x \end{aligned}$$

$$= \frac{1}{2}\left(\begin{aligned} &f(x_0) + f(x_1) + f(x_1) + f(x_2) \\ &+ f(x_2) + f(x_3) + f(x_3) + f(x_4) \end{aligned}\right)\Delta x$$

$$= \frac{1}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))\Delta x$$

We can extend this analysis to n trapezoids.

$$T_n = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n))$$

Where $x_i = a + i\Delta x$

$\Delta x = \frac{b-a}{n}$

MA 16010 LESSON 27: NUMERICAL INTEGRATION (EXAMPLES)

Example 1: **Use the Trapezoid Rule to approximate $\int_0^3 x^2\,dx$ using $n = 3$. Round your answer to the nearest tenth.**

Solution: (1) First calculate Δx.

$a = 0$

$b = 3$

$b - a = 3$

$\Delta x = \dfrac{b-a}{n} = \dfrac{3}{3} = 1$

(2) Determine what $f(x)$ is.

$$\int_0^3 \boxed{x^2}\,dx$$

Hence $f(x) = x^2$

(3) Find the following values:

$x_0 = 0$	$f(x_0) = 0^2 = 0$	$f(x_0) = 0$
$x_1 = 1$	$f(x_1) = 1^2 = 1$	$2 \cdot f(x_1) = 2 \cdot 1 = 2$
$x_2 = 2$	$f(x_2) = 2^2 = 4$	$2 \cdot f(x_2) = 2 \cdot 4 = 8$
$x_3 = 3$	$f(x_3) = 3^2 = 9$	$f(x_3) = 9$

(4) Sum all the values in the black box. $= 19$

(5) Multiply the value found in (4), Δx found in (1), and 1/2, which yields our answer.

$9 \cdot 1 \cdot \frac{1}{2} = 9/2$

MA 16010 LESSON 27: NUMERICAL INTEGRATION (EXAMPLES)

Example 2: Use the Trapezoid Rule to approximate $\int_2^5 ln(x+3)\, dx$ using $n = 3$. Round your answer to the nearest hundredth.

Solution: (1) First calculate Δx.

$a = 2$

$b = 5$

$b - a = 3$

$\Delta x = \dfrac{b-a}{n} = \dfrac{3}{3} = 1$

(2) Determine what $f(x)$ is.

$$\int_2^5 \boxed{\ln(x^2+3)}\, dx$$

Hence $f(x) = \ln(x^2+3)$

(3) Find the following values:

$x_0 = 2$	$f(x_0) = \ln(2^2+3)$	$f(x_0)$	$= \ln(7)$
$x_1 = 3$	$f(x_1) = \ln(3^2+3)$	$2 \cdot f(x_1)$	$= 2\ln(12)$
$x_2 = 4$	$f(x_2) = \ln(4^2+3)$	$2 \cdot f(x_2)$	$= 2\ln(19)$
$x_3 = 5$	$f(x_3) = \ln(5^2+3)$	$f(x_3)$	$= \ln(28)$

(4) Sum all the values in the black box. ≈ 16.1368

(5) Multiply the value found in (4), Δx found in (1), and 1/2, which yields our answer.

$16.1368 \times 1 \times \frac{1}{2} \approx 8.07$

MA 16010 LESSON 27: NUMERICAL INTEGRATION (EXAMPLES)

Example 3: **Use the Trapezoid Rule to approximate $\int_4^{12} \sqrt{x^3+7}\, dx$ using $n = 4$. Round your answer to the nearest hundredth.**

Solution: (1) First calculate Δx.

$a = 4$

$b = 12$

$b - a = 8$

$\Delta x = \dfrac{b-a}{n} = \dfrac{8}{4} = 2$

(2) Determine what $f(x)$ is.

$$\int_4^{12} \boxed{\sqrt{x^3+7}}\, dx$$

Hence $f(x) = x^3 + 7$

(3) Find the following values:

$x_0 = 4$	$f(x_0) = \sqrt{4^3+7}$	$f(x_0) = \sqrt{71}$
$x_1 = 6$	$f(x_1) = \sqrt{6^3+7}$	$2 \cdot f(x_1) = 2\sqrt{223}$
$x_2 = 8$	$f(x_2) = \sqrt{8^3+7}$	$2 \cdot f(x_2) = 2\sqrt{519}$
$x_3 = 10$	$f(x_3) = \sqrt{10^3+7}$	$2 \cdot f(x_3) = 2\sqrt{1007}$
$x_4 = 12$	$f(x_4) = \sqrt{12^3+7}$	$f(x_4) = \sqrt{1735}$

(4) Sum all the values in the black box. ≈ 188.9755

(5) Multiply the value found in (4), Δx found in (1), and 1/2, which yields our answer.

$188.9755 \cdot 2 \cdot \frac{1}{2} \approx 189$

MA 16010 LESSON 27: NUMERICAL INTEGRATION (EXAMPLES)

<u>Example 4:</u> Approximate the area of the shaded region by using the Trapezoid Rule with $n = 3$

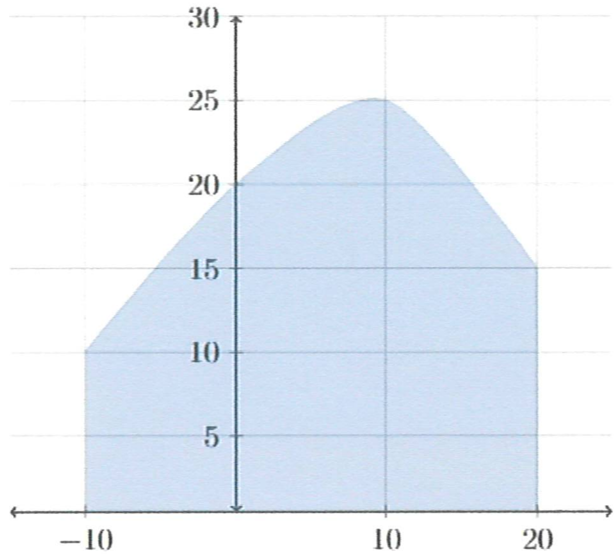
Solution: (1) First calculate Δx.

$a = -10$

$b = 20$

$b - a = 30$

$\Delta x = \dfrac{b-a}{n} = \dfrac{30}{3} = 10$



(2) Find the following values:

$x_0 =$	-10	$f(x_0) =$	10	$f(x_0) =$	10
$x_1 =$	0	$f(x_1) =$	20	$2 \cdot f(x_1) =$	40
$x_2 =$	10	$f(x_2) =$	25	$2 \cdot f(x_2) =$	50
$x_3 =$	20	$f(x_3) =$	15	$f(x_3) =$	15

(3) Sum all the values in the black box. $= 115$

(4) Multiply the value found in (3), Δx found in (1), and 1/2, which yields our answer.

$115 \times 10 \times \frac{1}{2} = 575$

MA 16010 LESSON 27: NUMERICAL INTEGRATION (EXAMPLES)

Example 5: Approximate the area of the shaded region by using the Trapezoid Rule with $n = 4$

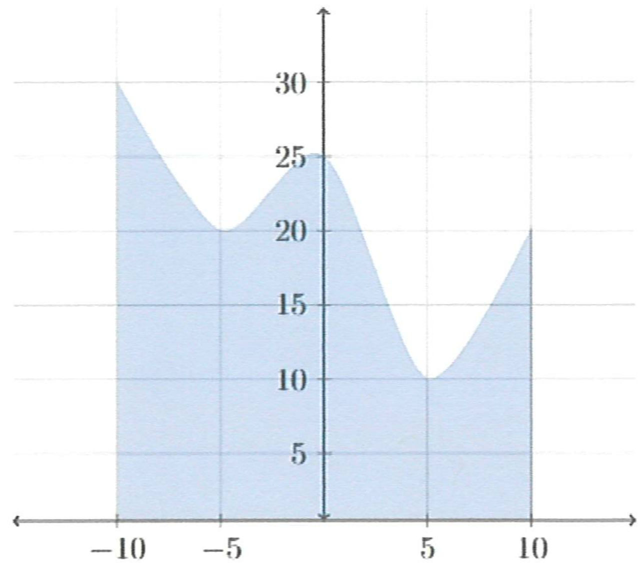
Solution: (1) First calculate Δx.

$a = -10$

$b = 10$

$b - a = 20$

$\Delta x = \dfrac{b-a}{n} = \dfrac{20}{4} = 5$



(2) Find the following values:

$x_0 = -10$	$f(x_0) = 30$	$f(x_0) = 30$
$x_1 = -5$	$f(x_1) = 20$	$2 \cdot f(x_1) = 40$
$x_2 = 0$	$f(x_2) = 25$	$2 \cdot f(x_2) = 50$
$x_3 = 5$	$f(x_3) = 10$	$2 \cdot f(x_3) = 20$
$x_4 = 10$	$f(x_4) = 20$	$f(x_4) = 20$

(3) Sum all the values in the black box. $= 160$

(4) Multiply the value found in (3), Δx found in (1), and 1/2, which yields our answer.

$160 \times 5 \times \frac{1}{2} = 400$