Lesson 28: Exponential Growth

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Ex 1: Solve IVP $\frac{dy}{dt} = 2y$ with $y(0) = 100$

Try getting all y terms to one side.

$$\frac{dy}{y} = 2dt$$

$$\int \frac{dy}{y} = \int 2dt$$

$$e^{\ln y} = e^{2t + C}$$

$$y = e^{2t+C}$$

$$= e^{2t} \cdot \underbrace{e^{C}}_{\text{constant}}$$

$$= Ce^{2t}$$

Ex 1: Solve IVP $\frac{dy}{dt} = 2y$ with $y(0) = 100$

Now with $y = Ce^{2t}$ use

$$100 = Ce^{2(0)} = C$$

So $\boxed{y = 100e^{2t}}$

Exponential Growth Model

If y is a differential function of t such that

$$\frac{dy}{dt} = y' = ky \quad \text{for some constant } k$$

then $y = Ce^{kt}$ where C is a constant.

k - proportionality constant or growth rate

C - the initial value of y

If $k > 0$ and $C > 0$, then this model is called the exponential growth model.

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Example 2: **The rate of change of a population P is proportional to P (use k for the proportionality constant). Answer the following questions.**

a) What is $\frac{dP}{dt}$?

$$\frac{dP}{dt} = kP$$

b) Find P.

$$P = Ce^{kt}$$

c) If $P = 200$ when $t = 0$ and $P = 400$ when $t = 2$, what is $P(4)$?

$$200 = Ce^{k(0)} \quad (1)$$
$$200 = C$$

$$400 = 200e^{2k} \quad (2)$$
$$2 = e^{2k}$$
$$\ln 2 = 2k$$
$$k = \frac{\ln(2)}{2}$$

$$P = 200\exp\left[\frac{\ln(2)}{2}t\right]$$

$$P(4) = 200\exp\left[\frac{\ln(2)}{2} \cdot 4\right] = 800$$

d) If $P = 200$ when $t = 1$ and $P = 400$ when $t = 2$, what is $P(4)$?

$$200 = Ce^{k} \quad (1)$$
$$400 = Ce^{2k} = \underbrace{Ce^{k}}e^{k} \quad (2)$$

Plug (1) into

$$400 = 200e^{k}$$
$$2 = e^{k}$$
$$\ln 2 = k$$

Plug k into (1)

$$200 = C\exp[\ln 2]$$
$$200 = C \cdot 2$$
$$C = 100$$

$$P = 100\exp[\ln(2) \cdot t]$$

$$P(4) = 100\exp[\ln(2) \cdot 4] = 1600$$

Example 3: **In a savings account where the interest is compounded continuously, if the initial investment is \$500 and the annual interest rate is 3%, how much money will there be in 10 years?**

Note $C = 500 > 0$ and $k = 0.03 > 0$. So

$$y = 500 \exp[0.03t]$$

$$y(10) = 500 \exp[0.03 \cdot 10] \approx \$674.93$$

How long does it take to double the initial investment?

Previously, we found $y = 500 \exp[0.03t]$

Double the initial investement $\Rightarrow y = 2(500)$.

So $2(\cancel{500}) = \cancel{500} \exp[0.03t]$

$$\ln 2 = 0.03t$$

$$t = \frac{\ln 2}{0.03} \approx 23.1 \text{ yrs}$$

Example 4: **In a savings account where the interest is compounded continuously, if the initial investment is \$100 and there are \$150 in 8 years, what is the annual interest rate?**

Solving for k.

k-interest rate >0 and $C=100>0 \Rightarrow y=100e^{kt}$

Also the question also states $y(8)=150$.

$$150=100e^{k(8)}$$

$$\frac{150}{100}=e^{8k}$$

$$\frac{3}{2}=e^{8k}$$

$$\ln\left(\frac{3}{2}\right)=8k$$

$$k=\frac{1}{8}\ln\left(\frac{3}{2}\right) \approx 0,05 \Rightarrow 5\%$$

Example 5: **Suppose you deposited \$15,000 in a saving account in which interest is compounded continuously. It takes 20 years to double your money in this account. What is the annual rate of interest?**

Solving for k.

k- interest rate >0 and $C=15{,}000 \Rightarrow y=15{,}000e^{kt}$

Also the question also states $y(20)=2(15{,}000)$

$$2(\cancel{15{,}000})=\cancel{15{,}000}e^{k(20)}$$

$$2=e^{k(20)}$$

$$\ln 2=k(20)$$

$$k=\frac{\ln(2)}{20}\approx 0.0347 \Rightarrow 3.47\%$$