Lesson 2: Finding Limits Analytically

There are 3 different cases to consider.

① $f(c)$ return a # (it could be 0)

i.e. $f(x)$ is continuous @ $x=c$.

i.e. $\lim_{x \to c} f(x) = f(c)$

Example 1: $\lim_{x \to 4} (2x-3) = 2(4)-3 = 8-3 = 5$

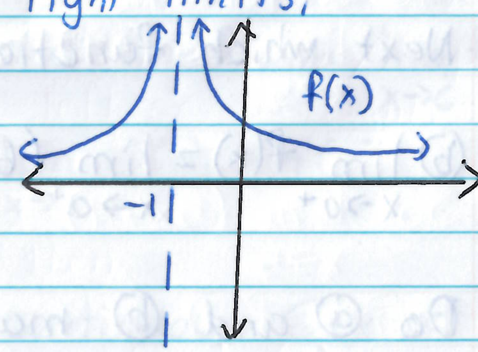
② $f(c)$ returns $\frac{\text{nonzero } \#}{0}$

i.e. $f(x)$ has a Vertical Asymptote @ $x=c$

i.e. $\lim_{x \to c} f(x) = \pm\infty$ or $=$ DNE

Example 2: $\lim_{x \to -1} \frac{1}{(x+1)^2}$

$f(-1) = \frac{1}{(-1+1)^2} = \frac{1}{0} \Rightarrow$ We need to check the left and right limits.



Based on the graph on the right,

$\lim_{x \to -1^-} \frac{1}{(x+1)^2} = \infty$

$\lim_{x \to -1^+} \frac{1}{(x+1)^2} = \infty$

and since both limits match

$\lim_{x \to -1} \frac{1}{(x+1)^2} = \infty$

③ $f(c)$ returns $\frac{0}{0}$

i.e. $f(x)$ has a hole (if a factor cancels out) or VA @ $x=c$ (no factor cancels)

Idea: Manipulate $f(x)$ so that it can look like case 1 or 2.

Example 3: $\lim_{x \to 3} \frac{x^3 - 3x^2}{x - 3}$

$f(3) = \frac{3^3 - 3(3)^2}{3-3} = \frac{27-27}{3-3} = \frac{0}{0} \Rightarrow$ Let's try factoring

$$\lim_{x \to 3} \frac{x^3 - 3x^2}{x - 3} = \lim_{x \to 3} \frac{x^2\cancel{(x-3)}}{\cancel{x-3}} = \lim_{x \to 3} x^2 = 3^2 = 9$$

Example 4: Let $f(x) = \begin{cases} 13x^2 - 6, & x \le 0 \\ 6x + 6, & x > 0 \end{cases}$

Find the following limits.

ⓐ $\lim_{x \to 0^-} f(x)$ ⓑ $\lim_{x \to 0^+} f(x)$ ⓒ $\lim_{x \to 0} f(x)$

Start by asking yourself which function is the left of 0? $f(x) = 13x^2 - 6$

ⓐ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (13x^2 - 6) = 13(0)^2 - 6 = -6$

Next which function is to the right of 0? $f(x) = 6x + 6$

ⓑ $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (6x + 6) = 6(0) + 6 = 6$

Do ⓐ and ⓑ match? No

ⓒ $\lim_{x \to 0} f(x) = \text{DNE}$

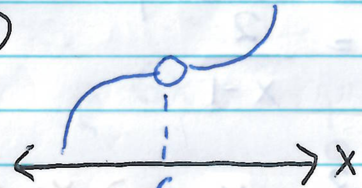
Continuity

Definition: A function is continuous if there is no disruption in the graph.

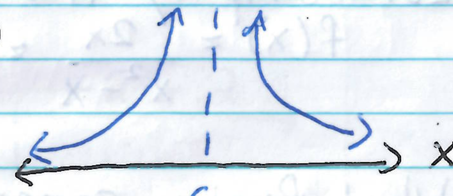
i.e. Does your pencil lift off the paper? No. Then it's continuous.

The following 4 graphs show $f(x)$ is discontinuous @ $x = c$.

①

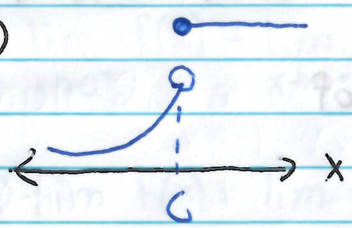


②



These 2 graphs have $f(c)$ undefined

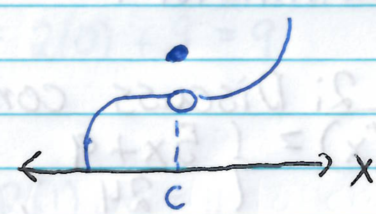
③



$f(c)$ is defined

BUT $\lim_{x \to c} f(x) = DNE$

④



$f(c)$ is defined and $\lim_{x \to c} f(x)$ exist

BUT $\lim_{x \to c} f(x) \neq f(c)$

Definition: A function, $f(x)$, is continuous @ $x = c$ if all the following are true:

① $f(c)$ defined (has a value)

② $\lim_{x \to c} f(x)$ exists

③ $\lim_{x \to c} f(x) = f(c)$

If any of the 3 conditions aren't met, then we say $f(x)$ is discontinuous @ $x = c$.

Example 1: Discuss continuity of $f(x) = \frac{2x}{x^2 - x}$

When is f(x) undefined? When denominator = 0

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \mid x - 1 = 0$

$x = 1$

Hence we have discontinuities @ $x=0$ and $x=1$

But what kind of discontinuity are they? Jump? Hole? VA?

Let's simplify f(x) to answer that question.

$$f(x) = \frac{2x}{x^2 - x} = \frac{2}{x(x-1)} = \frac{2}{x-1}$$

What factor canceled out? $x \Rightarrow x = 0 \Rightarrow$ Hole

What factor remains in the denominator? $x - 1 \Rightarrow x = 1 \Rightarrow$ VA

Example 2: Discuss continuity of

$$f(x) = \begin{cases} 5x+9, & x \neq 5 \\ 24, & x = 5 \end{cases}$$

To determine continuity, we need to check

$\lim_{x \to 5} (5x+9) = 24$

So $\lim_{x \to 5} (5x+9) = 5(5)+9 = 34 \neq 24$

Hence f(x) has a hole @ $x = 5$.

How? Graph f(x) and you see a straight line w/ a hole

Example 3: Discuss continuity of

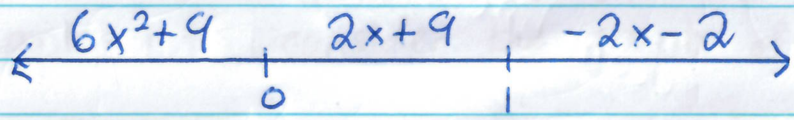
$$f(x) = \begin{cases} 6x^2+9 & \text{if } x \le 0 \\ 2x+9 & \text{if } 0 < x < 1 \\ -2x-2 & \text{if } x \ge 1 \end{cases}$$

To determine continuity, we need to check

① $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$

② $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$

Let's arrange each piece of the function on a # line,



① $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (6x^2+9) = 6(0)^2+9 = 9$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (2x+9) = 2(0)+9 = 9$

$\|$ $\Rightarrow$ No discontinuity @ $x=0$

② $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (2x+9) = 2(1)+9 = 11$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (-2x-2) = -2(1)-2 = -4$

$\nparallel$ $\Rightarrow$ Discontinuity @ $x=1$

$\Rightarrow$ Jump @ $x=1$