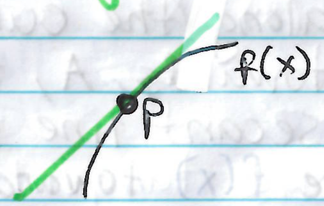
Lesson 3: The Derivative

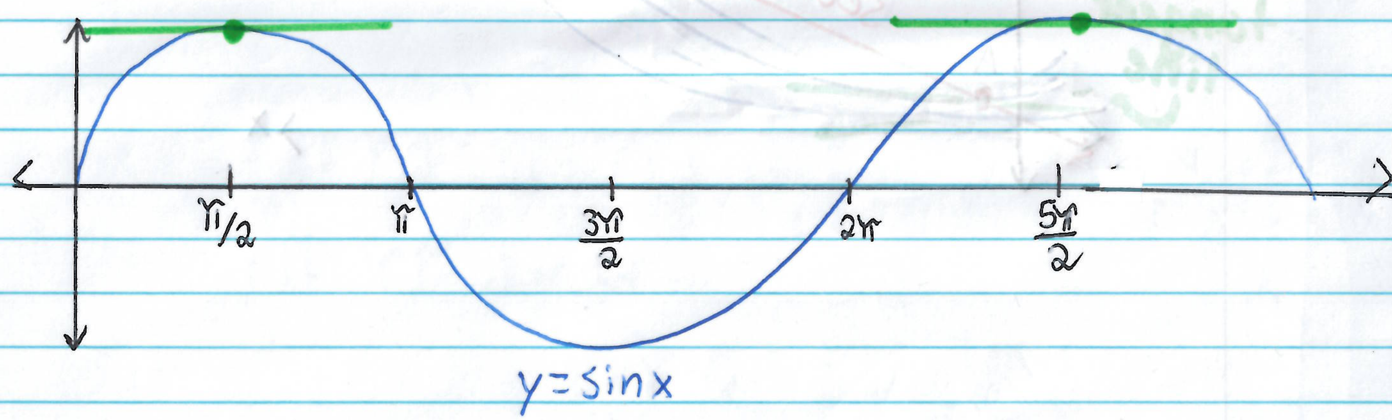
Before we define the derivative, let's define a tangent line.

Let $f(x)$ be the following curve in black. Then **tangent line is the green line.** Also let P be a point where both intersect.



From the image, we can say the tangent line is a straight line that touches $f(x)$ at point P, but does not cross P.

The issue with this definition doesn't work all the time. For example, let's consider the graph of $y = \sin x$.

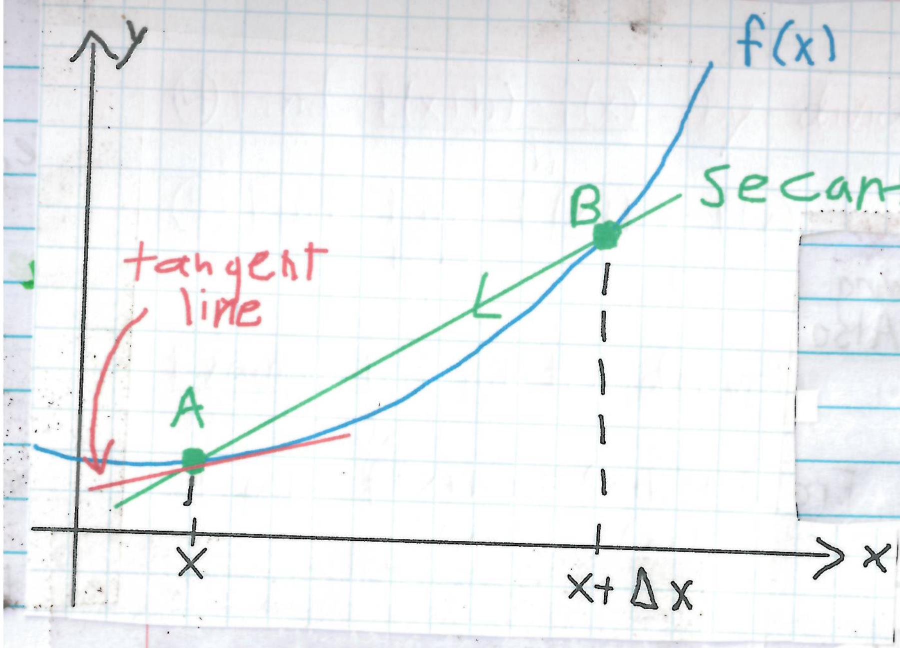


Let **the green lines** represent the tangent lines at $x = \pi/2$, and $x = 5\pi/2$, respectively.

If we extend each **green line** we see that they will overlap Hence overlap/cross the function, $y = \sin x$, twice.

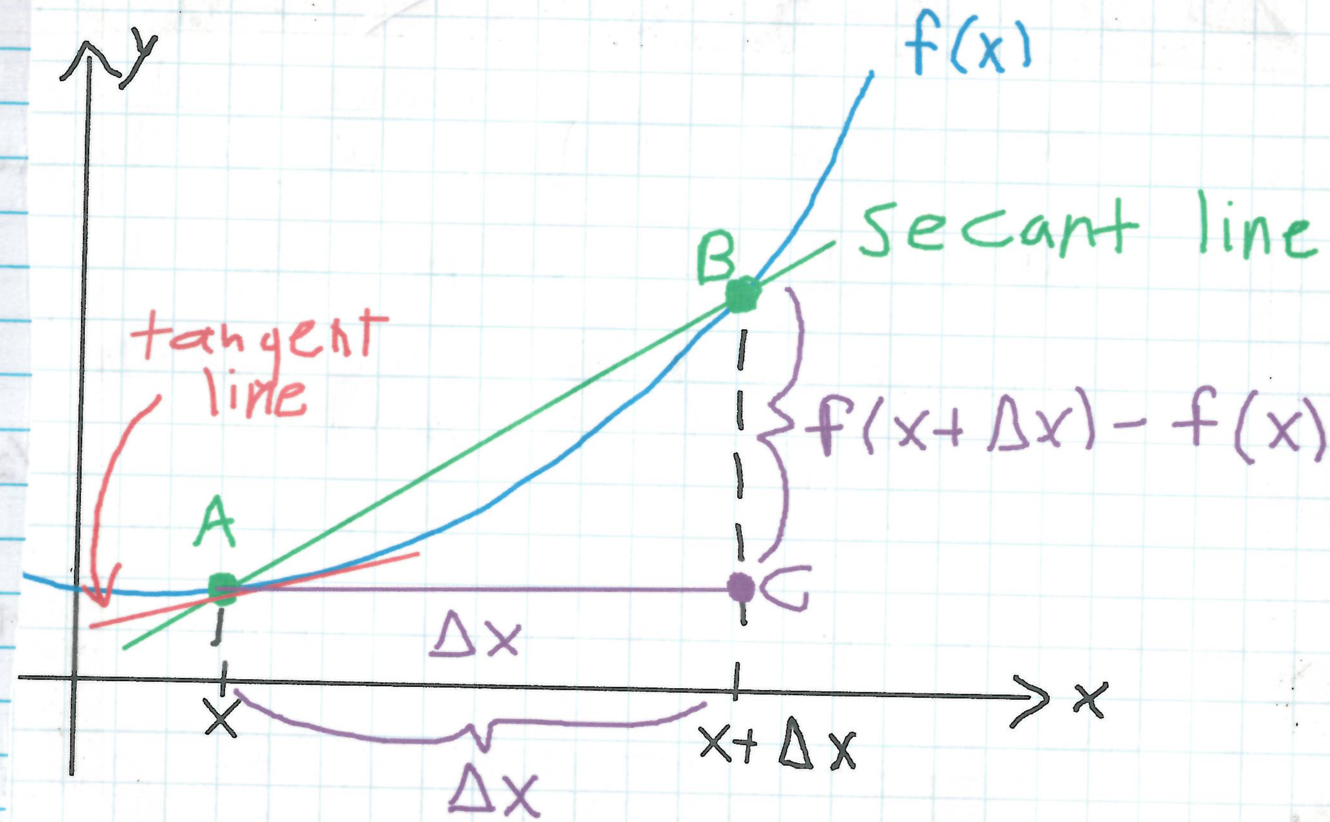
We want a precise definition for all scenarios. Before we do so, we need to introduce the secant line.

<u>Definition:</u> A secant line of $f(x)$ is a straight line that goes through 2 distinct points of $f(x)$.



Note if we move B along the curve f(x), towards A, then the secant line becomes the tangent line.

Now, let's determine the secant line's slope.



Hence we can say the slope of the secant line is

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

But we want to know more about the tangent line, not the secant line. So how can we achieve that?

If we knew the slope of the tangent line, then we can use the point-slope formula to find it's equation.

Recall the point-slope formula:
Given m - slope and (x_1, y_1) - point,

$$y - y_1 = m(x - x_1)$$

Again the secant line becomes the tangent line as on ...point gets closer to the other, So

$$\text{Slope of Tangent Line} = \lim_{\Delta x \to 0} \text{Slope of Secant Line} = \lim_{\Delta x \to 0} \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{Also known as Difference Quotient}}$$

How is the slope of the tangent line related to the derivative? They are the same!

Definition: The derivative of $f(x)$ @ x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{where } h = \Delta x$$

Different Notation: y', $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}[f(x)]$

Game Plan: To find the derivative using the limit definition, follow the following steps:
① Find $f(x+h)$
② Find $-f(x)$
③ Find $f(x+h) - f(x)$ by adding ① + ②

④ Find $\dfrac{f(x+h)-f(x)}{h}$ by dividing ③ by h.

⑤ Find $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ by taking $\lim\limits_{h\to 0}$ of ④

Example 1: Find the derivative of $f(x) = x+5$ using limit definition.

Step 1: $f(x+h) = (x+h)+5 = \cancel{x} + h + \cancel{5}$

Step 2: $-f(x) = -(x+5) = \cancel{-x} \cancel{-5}$

$$\overline{h}$$

Step 3: Add ①+②

Step 4: Divide ③ by h: $\dfrac{h}{h} = 1$

Step 5: Take $\lim\limits_{h\to 0}$ of ④: $\lim\limits_{h\to 0} 1 = 1 = f'(x)$

Useful formulas

- Perfect Square: $(a \pm b)^2 = a^2 \pm 2ab + b^2$
- Difference of Squares: $a^2 - b^2 = (a-b)(a+b)$

Example 2: The derivative of a function is found by

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^3+2} - \sqrt{x^3+2}}{h}$$

What is $f(x)$?

Recall $\lim\limits_{h\to 0} \dfrac{f(x+h) - \boxed{f(x)}}{h} = f'(x)$. It is enough to focus our attention to $\boxed{}$ to determine $f(x)$. So

$$f(x) = \sqrt{x^3+2}$$

Example 3: Given $f(x) = x^2 - 3$.

ⓐ Find the slope of the tangent line, i.e. Find $f'(x)$.

Step 1: $f(x+h) = (x+h)^2 - 3 = x^2 + 2xh + h^2 - 3$
Step 2: $-f(x) = -(x^2 - 3) = -x^2 + 3$
Step 3: Add ① + ② $\quad 2xh + h^2$
Step 4: Divide ③ by h: $\dfrac{2xh+h^2}{h} = \dfrac{h(2x+h)}{h} = 2x+h$

Step 5: Take $\lim\limits_{h\to 0}$ of ④: $\lim\limits_{h\to 0}(2x+h) = 2x+0 = 2x = f'(x)$

But how do I find the equation of the tangent line?
Game Plan: To find the equation of the tangent line to f(x) at the point x=c, follow the following steps:
① Find $f'(x)$
② Calculate $f'(c)$
③ Calculate $f(c)$
④ Plug ② + ③ into the point-slope formula: $y - f(c) = f'(c)(x-c)$

Example 3: Given $f(x) = x^2 - 3$
ⓑ Find the equation of the tangent line to f(x) at x=2.
Step 1: $f'(x) = 2x$ (from part a)
Step 2: $f'(2) = 2(2) = 4$
Step 3: $f(2) = 2^2 - 3 = 4 - 3 = 1$
Step 4: Plug ② + ③ into the point-slope formula

$$y - f(2) = f'(2)(x-2)$$
$$y - 1 = 4(x-2)$$
$$y - 1 = 4x - 8$$
$$+1 \qquad\quad +1$$
$$y = 4x - 7$$