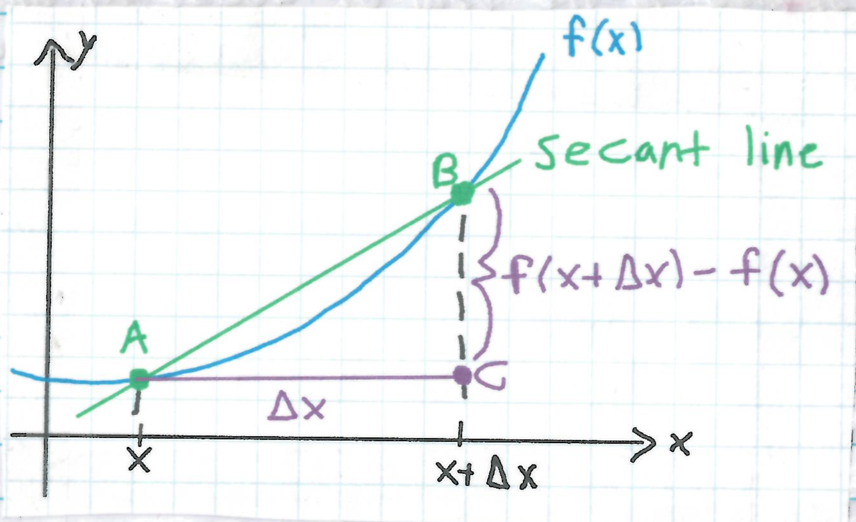
Lesson 5: Instantaneous Rates of Change



Recall this image from Lesson 3.

We found from here that the slope of the secant line is $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

This quantity is also called average rate of change.

In addition, when average rate of change approaches a quantity is called instantaneous rate of change.

i.e. Instantaneous rate of change is the derivative.

Example 1: The initial population of a culture of bacteria is 1000. The population after t hours, $P(t)$, is given by

$$P(t) = 2t^2 + 8t + 1000$$

ⓐ Find the number of bacteria present after 5 hours.

i.e. What's $P(5)$?

$$P(5) = 2(5)^2 + 8(5) + 1000 = 1090$$

ⓑ Find the rate of change of the population after 5 hours.

i.e. What's $P'(5)$?

$$P'(t) = 4t + 8$$

$$P'(5) = 4(5) + 8 = 28$$

Example 2: The population of a city since the year 2000 can be modeled by

$$P(t) = 500t^2 - 400t + 20000$$

where $t = 0$ corresponds to the year of 2000. In which year is the population increasing at the rate of 8600 people per year?

i.e. Solve $P'(t) = 8600$ for t.

$$P'(t) = 1000t - 400 = 8600$$
$$1000t = 9000$$
$$t = 9 \quad \Rightarrow 2009$$

Position & Velocity Functions

Position Function, $s(t)$, tells us how far away an object is.

Velocity Function, $v(t)$, tells us speed of an object with respect to direction.

To find Velocity we take the derivative of the position i.e. $v(t) = s'(t)$

Example 3: An object is shot upward from the surface of Earth. The position function is

$$s(t) = -4.9t^2 + 98t$$

ⓐ Find $v(t)$.

$$v(t) = s'(t) = -4.9(2)t + 98 = -9.8t + 98$$

ⓑ Find $v(3)$

$$v(3) = -9.8(3) + 98 = 68.6$$

ⓒ What is the velocity of the object when it hits the ground?

i.e. Solve $s(t) = 0$ for t. Then plug t into $v(t)$.

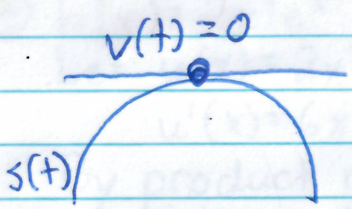
$$0 = s(t) = -4.9t^2 + 98t$$
$$0 = -4.9t(t - 20)$$

$$-4.9t = 0 \quad\Big|\quad t - 20 = 0$$
$$t = 0 \quad\Big|\quad t = 20$$

We ignore $t=0$ b/c that is before it's through. So plug $t=20$ into $v(t)$.

$$v(20) = -9.8(20) + 98 = -98$$

ⓓ When is the object at its highest point?



i.e. Solve $v(t)=0$ for t.

$$-9.8t + 98 = 0$$
$$98 = 9.8t$$
$$\frac{98}{9.8} = t$$
$$t = 10$$

Example 4: Let $C = 2\pi r$. What is the rate of change of C with respect to r?

i.e. Find $\frac{dC}{dr}$

$$\frac{d}{dr}[C] = \frac{d}{dr}[2\pi r]$$

$$\frac{d}{dr}[C] = 2\pi \frac{d}{dr}[r]$$

$$1 \cdot \frac{dC}{dr} = 2\pi \cancel{\frac{dr}{dr}}^{\,1}$$

$$\frac{dC}{dr} = 2\pi$$

Example 5: Let $p = 3q - 5$

ⓐ What is the rate of change of p with respect to q?

i.e. Find $\frac{dp}{dq}$

$$\frac{d}{dq}[p] = \frac{d}{dq}[3q-5]$$

$$\frac{d}{dq}[p] = 3\frac{d}{dq}[q] - \frac{d}{dq}[5]$$

$$1 \cdot \frac{dp}{dq} = 3\,\cancel{\frac{dq}{dq}}^{\,1} - 0$$

$$\frac{dp}{dq} = 3$$

ⓑ What is the rate of change of q with respect to p? i.e. Find $\frac{dq}{dp}$

$$\frac{d}{dp}[p] = \frac{d}{dp}[3q-5]$$

$$\frac{d}{dp}[p] = 3\frac{d}{dp}[q] - \frac{d}{dp}[5]$$

$$1 \cdot \cancel{\frac{dp}{dp}}^{\,1} = 3\frac{dq}{dp} - 0$$

$$1 = 3\frac{dq}{dp}$$

$$\frac{1}{3} = \frac{dq}{dp}$$