Lesson 6: The Product Rule

Product Rule: Let $h(x) = u(x)\,v(x)$. Then

$$h'(x) = \frac{d}{dx}[u(x)v(x)]$$
$$= \frac{d}{dx}[u(x)] \cdot v(x) + u(x)\frac{d}{dx}[v(x)]$$
$$= u'(x)\,v(x) + u(x)\,v'(x)$$

Example 1: Find the derivative of the following functions:

ⓐ $h(x) = 2x^3e^x$

Let $u(x) = 2x^3$ $\quad$ $v(x) = e^x$

$u'(x) = 6x^2$ $\quad$ $v'(x) = e^x$

By product rule,

$$h'(x) = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= 6x^2e^x + 2x^3e^x$$

ⓑ $h(x) = 4e^x\sin x$

Let $u(x) = 4e^x$ $\quad$ $v(x) = \sin x$

$u'(x) = 4e^x$ $\quad$ $v'(x) = \cos x$

By product rule,

$$h'(x) = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= 4e^x\sin x + 4e^x\cos x$$

ⓒ $h(x) = \sqrt{x}\,(2x^2+4)$

Method 1: Find $h'(x)$ by product rule

Let $u(x) = \sqrt{x} = x^{1/2}$ $\quad$ $v(x) = 2x^2+4$

$u'(x) = \frac{1}{2}x^{-1/2}$ $\quad$ $v'(x) = 4x$

By product rule,

$$h'(x) = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= \frac{1}{2}x^{-1/2}(2x^2+4) + x^{1/2}(4x)$$
$$= x^{3/2} + 2x^{-1/2} + 4x^{3/2}$$
$$= 5x^{3/2} + 2x^{-1/2}$$

Method 2: Expand $h(x)$ to use Power Rule

$$h(x) = x^{1/2}(2x^2+4)$$
$$= 2x^{5/2} + 4x^{1/2}$$

By power rule,

$$h'(x) = 2\left(\frac{5}{2}\right)x^{3/2} + 4\left(\frac{1}{2}\right)x^{-1/2}$$
$$= 5x^{3/2} + 2x^{-1/2}$$

Moral: Just b/c there is a product doesn't mean you need to use the product rule.

ⓓ $h(x) = (x^2+5x)(-3x^5+6)$

Let's expand $h(x)$.

	$-3x^5$	6
x^2	$-3x^7$	$6x^2$
$5x$	$-15x^6$	$30x$

$\Rightarrow h(x) = -3x^7 - 15x^6 + 6x^2 + 30x$

By power rule,

$$h'(x) = -3(7)x^6 - 15(6)x^5 + 6(2)x + 30$$
$$= -21x^6 - 90x^5 + 12x + 30$$

And sometimes you have only product rule to use ... at least until we learn Chain Rule.

ⓔ $h(x) = 4\sin x\,(7\cos x + 6\sin x)$

Let $u(x) = 4\sin x$ $\quad v(x) = 7\cos x + 6\sin x$

$u'(x) = 4\cos x$ $\quad v'(x) = -7\sin x + 6\cos x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 4\cos x\,(7\cos x + 6\sin x) + 4\sin x\,(-7\sin x + 6\cos x)$$
$$= 28\cos^2 x + 24\cos x\sin x - 28\sin^2 x + 24\sin x\cos x$$
$$= 28\cos^2 x + 48\sin x\cos x - 28\sin^2 x$$

Example 2: Let $h(x) = x^2 \sin x$. Compute $h'(\pi/6)$

Let $u(x) = x^2$ $\quad v(x) = \sin x$

$u'(x) = 2x$ $\quad v'(x) = \cos x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 2x\sin x + x^2\cos x$$

To find $h'(\pi/6)$, we need the trig table from Quiz 1.

$$h'\left(\frac{\pi}{6}\right) = 2\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + \left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\pi}{3} \cdot \frac{1}{2} + \frac{\pi^2}{36} \cdot \frac{\sqrt{3}}{2}$$
$$= \frac{\pi}{6} + \frac{\pi^2\sqrt{3}}{72}$$

Example 3: Find all the x-values at which $y = 4x^6e^x$ has a horizontal tangent.

Recall a horizontal line has a slope of 0. So a horizontal tangent means $y'(x) = 0$.

i.e. We want to solve $y'(x) = 0$ for x.

Let $u(x) = 4x^6$ $\quad v(x) = e^x$

$u'(x) = 24x^5$ $\quad v'(x) = e^x$

By product rule,

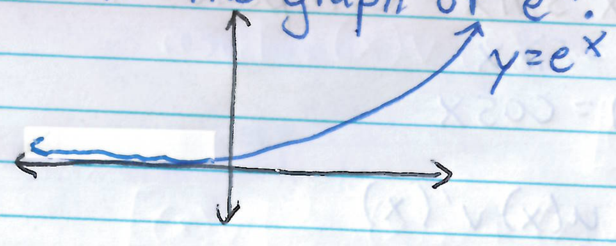
$$y'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 24x^5e^x + 4x^6e^x$$

So $y'(x) = 0$

$$24x^5e^x + 4x^6e^x = 0$$
$$4x^5e^x(6+x) = 0$$

$4x^5 = 0$	$e^x = 0$	$6 + x = 0$
$x = 0$	$\downarrow$ Never happens	$x = -6$

Recall the graph of e^x.



Does the graph cross/touch the x-axis? No!

Hence our final answer is $x=0$, $x=-6$.