Lesson 7: Quotient Rule

Quotient Rule: Let $h(x) = \frac{u(x)}{v(x)}$. Then

$$h'(x) = \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right]$$
$$= \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

Example 1: Find the derivative of the following functions:

ⓐ $h(x) = \frac{1}{x^2}$

Method 1: Use Power Rule

Rewrite $h(x)$: $h(x) = x^{-2}$

By power rule, $h'(x) = -2x^{-3} = -\frac{2}{x^3}$

Method 2: Use Quotient Rule

Let $u(x) = 1$ $\quad v(x) = x^2$

$u'(x) = 0$ $\quad v'(x) = 2x$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$
$$= \frac{0 \cdot x^2 - 1(2x)}{(x^2)^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

ⓑ $h(x) = \frac{x^3}{x^2+1}$

Let $u(x) = x^3$ $\quad v(x) = x^2+1$

$u'(x) = 3x^2$ $\quad v'(x) = 2x$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$
$$= \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2}$$

$$= \frac{3x^4 + 3x^2 - 2x^4}{(x^2+1)^2} = \frac{x^4 + 3x^2}{(x^2+1)^2}$$

ⓒ $h(x) = \dfrac{x-3}{6\sqrt{x}+7}$

Let $u(x) = x-3$ $\qquad v(x) = 6\sqrt{x} + 7 = 6x^{1/2} + 7$

$u'(x) = 1$ $\qquad v'(x) = 6\left(\frac{1}{2}\right)x^{-1/2} = 3x^{-1/2}$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{1 \cdot (6x^{1/2}+7) - (x-3)(3x^{-1/2})}{(6x^{1/2}+7)^2}$$

$$= \frac{6x^{1/2} + 7 - 3x^{1/2} + 9x^{-1/2}}{(6x^{1/2}+7)^2}$$

$$= \frac{3x^{1/2} + 7 + 9x^{-1/2}}{(6x^{1/2}+7)^2}$$

ⓓ $h(x) = \dfrac{\sin x}{6x+9}$

Let $u(x) = \sin x$ $\qquad v(x) = 6x+9$

$u'(x) = \cos x$ $\qquad v'(x) = 6$

By quotient rule,

$$h'(x) = \frac{u'(x)\,v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{(\cos x)(6x+9) - (\sin x)6}{(6x+9)}$$

$$= \frac{(6x+9)\cos x - 6\sin x}{(6x+9)^2}$$

ⓔ $h(x) = \dfrac{e^x}{6-e^x}$

Let $u(x) = e^x$ $\qquad v(x) = 6 - e^x$

$u'(x) = e^x$ $\qquad v'(x) = -e^x$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{e^x(6-e^x) - e^x(-e^x)}{(6-e^x)^2}$$

$$= \frac{6e^x - (e^x)^2 + (e^x)^2}{(6-e^x)^2}$$

$$= \frac{6e^x}{(6-e^x)^2}$$

Derivatives of Other Trig Functions

Recall from Lesson 5,

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Also recall from our trig days,

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Using quotient rule on these expressions do we yield

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Example 2: Find the derivative of the following function:

ⓐ $h(x) = 3\sin x \tan x$

Let $u(x) = 3\sin x \qquad v(x) = \tan x$

$u'(x) = 3\cos x \qquad v'(x) = \sec^2 x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$

$$= 3\cos x \tan x + 3\sin x \sec^2 x$$

$$= 3\cancel{\cos x}\frac{\sin x}{\cancel{\cos x}} + 3\sin x\left(\frac{1}{\cos x}\right)$$

$$= 3\sin x + 3\sin x \cos^2 x$$

(b) $h(x) = 2e^x \csc x$

Let $u(x) = 2e^x$ $\quad v(x) = \csc x$

$u'(x) = 2e^x$ $\quad v'(x) = -\csc x \cot x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 2e^x \csc x + 2e^x(-\csc x \cot x)$$
$$= 2e^x \csc x - 2e^x \csc x \cot x$$

(c) $h(x) = 6 \sec x \tan x$

Let $u(x) = 6\sec x$ $\quad v(x) = \tan x$

$u'(x) = 6 \sec x \tan x$ $\quad v'(x) = \sec^2 x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= (6 \sec x \tan x)(\tan x) + (6 \sec x)(\sec^2 x)$$
$$= 6 \sec x \tan^2 x + 6 \sec^3 x$$

(d) $h(x) = x^2 \cot x$

Let $u(x) = x^2$ $\quad v(x) = \cot x$

$u'(x) = 2x$ $\quad v'(x) = -\csc^2 x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 2x \cot x + x^2(-\csc^2 x)$$
$$= 2x \cot x - x^2 \csc^2 x$$