Lesson 8: The Chain Rule Pt 1

Recall the Composition of Functions,

Let $y = f(g(x)) \Rightarrow \begin{cases} g \text{ - inner function} \\ f \text{ - outer function} \end{cases}$

Example 1: Determine the inner, g, and outer, f, function for the functions:

ⓐ $y = (3x+1)^2$

$f(x) = x^2 \qquad g(x) = 3x+1$

Check $y = f(g(x)) = f(3x+1) = (3x+1)^2$ ✓

ⓑ $y = \sin^2 x = (\sin x)^2$

$f(x) = x^2 \qquad g(x) = \sin x$

Check $y = f(g(x)) = f(\sin x) = (\sin x)^2$ ✓

ⓒ $y = \sin(x^2)$

$f(x) = \sin x \qquad g(x) = x^2$

Check $y = f(g(x)) = f(x^2) = \sin(x^2)$ ✓

ⓓ $y = \ln(2x)$

$f(x) = \ln x \qquad g(x) = 2x$

Check $y = f(g(x)) = f(2x) = \ln(2x)$ ✓

ⓔ $y = \sqrt[3]{2x+1}$

$f(x) = \sqrt[3]{x} \qquad g(x) = 2x+1$

Check $y = f(g(x)) = f(2x+1) = \sqrt[3]{2x+1}$ ✓

Chain Rule: Let $h(x) = u(v(x))$. Then

$$h'(x) = \frac{d}{dx}[u(v(x))]$$
$$= u'(v(x)) \cdot v'(x)$$

Example 2: Find the derivative of the following functions:

ⓐ $y = (3x+1)^2$

Let $u(x) = x^2$ $\quad v(x) = 3x+1$

$u'(x) = 2x$ $\quad v'(x) = 3$

By Chain Rule

$$y' = u'(\ (x))\ '(x)$$
$$= u'(3x+1) \cdot 3$$
$$= 2(3x+1) \cdot 3$$
$$= 6(3x+1)$$
$$= 18x + 6$$

ⓑ $y = 2\cos^3 x = 2(\cos x)^3$

Let $u(x) = 2x^3$ $\quad v(x) = \cos x$

$u'(x) = 6x^2$ $\quad v'(x) = -\sin x$

By Chain Rule,

$$y' = u'(\ (x))\ '(x)$$
$$= u'(\cos x) \cdot (-\sin x)$$
$$= 6(\cos x)^2 \cdot (-\sin x)$$
$$= -6\cos^2 x \sin x$$

ⓒ $y = \sqrt[5]{13+4x^8} = (13+4x^8)^{1/5}$

Let $u(x) = x^{1/5}$ $\quad v(x) = 13+4x^8$

$u'(x) = \frac{1}{5}x^{-4/5}$ $\quad v'(x) = 32x^7$

By Chain Rule,

$$y' = u'(v(x)) \cdot v'(x)$$
$$= u'(13+4x^8) \cdot (32x^7)$$
$$= \frac{1}{5}(13+4x^8)^{-4/5}(32x^7)$$
$$= \frac{32x^7}{5(13+4x^8)^{4/5}}$$

(d) $y = 6(3e^x - 27)^8$

Let $u(x) = 6x^8$ $\quad v(x) = 3e^x - 27$

$u'(x) = 48x^7$ $\quad v'(x) = 3e^x$

By Chain Rule,

$$\begin{aligned} y' &= u'(v(x))\,v'(x) \\ &= u'(3e^x - 27) \cdot (3e^x) \\ &= 48(3e^x - 27)^7 (3e^x) \\ &= 144e^x (3e^x - 27)^7 \end{aligned}$$

(e) $y = \dfrac{15}{\sqrt[3]{x^2+1}} = 15(x^2+1)^{-1/3}$

Let $u(x) = 15x^{-1/3}$ $\quad v(x) = x^2 + 1$

$u'(x) = 15\left(\dfrac{-1}{3}\right)x^{-4/3} = -5x^{-4/3}$ $\quad v'(x) = 2x$

By Chain Rule,

$$\begin{aligned} y' &= u'(v(x))\,v'(x) \\ &= u'(x^2+1) \cdot (2x) \\ &= -5(x^2+1)^{-4/3}(2x) \\ &= -\frac{10x}{(x^2+1)^{4/3}} \end{aligned}$$

(f) $y = \left(\dfrac{2x}{3x^2 + x}\right)^3$

Note you can do this problem by applying Chain Rule and then quotient rule, but things will get messy very fast.

Instead let's first rewrite y to get something more managable.

Rewrite y.

$$y = \left(\frac{2\cancel{x}}{\cancel{x}(3x+1)}\right)^3 = \left(\frac{2}{3x+1}\right)^3 = \frac{2^3}{(3x+1)^3} = 8(3x+1)^{-3}$$

Let $u(x) = 8x^{-3}$ $\qquad$ $v(x) = 3x+1$

$u'(x) = 8(-3)x^{-4} = -24x^{-4}$ $\qquad$ $v'(x) = 3$

By Chain Rule,

$$
\begin{aligned}
y' &= u'(v(x)) \cdot v'(x) \\
&= u'(3x+1) \cdot 3 \\
&= -24(3x+1)^{-4} \cdot 3 \\
&= -72(3x+1)^{-4} \\
&= \frac{-72}{(3x+1)^4}
\end{aligned}
$$