Lesson 9: The Chain Rule Pt 2

Recall from last time,

Chain Rule: Let $h(x) = u(v(x))$. Then
$$h'(x) = u'(v(x)) \cdot v'(x)$$

Example 1: Find the derivative of the following functions:

ⓐ $y = (5x-2)^3$

Let $u(x) = x^3$ $\quad v(x) = 5x-2$

$u'(x) = 3x^2$ $\quad v'(x) = 5$

By Chain Rule,
$$\begin{aligned} y &= u'(v(x)) \cdot v'(x) \\ &= u'(5x-2) \cdot 5 \\ &= 3(5x-2)^2 \cdot 5 \\ &= 15(5x-2)^2 \end{aligned}$$

ⓑ $y = 5\sin(x^2)$

Let $u(x) = 5\sin x$ $\quad v(x) = x^2$

$u'(x) = 5\cos x$ $\quad v'(x) = 2x$

By Chain Rule,
$$\begin{aligned} y &= u'(v(x)) \cdot v'(x) \\ &= u'(x^2) \cdot (2x) \\ &= 5\cos(x^2) \cdot (2x) \\ &= 10x\cos(x^2) \end{aligned}$$

ⓒ $y = e^{(1-2x)^4} = \exp[(1-2x)^4]$

Note to find the derivative of this function, we need to apply Chain Rule twice.

Let $u(x) = e^x$ $\quad v(x) = (1-2x)^4$

$a(x) = x^4$ $\quad b(x) = 1-2x$

$a'(x) = 4x^3$ $\quad b'(x) = -2$

$u'(x) = e^x$
$$\begin{aligned} v'(x) &= a'(b(x)) \cdot b'(x) \\ &= a'(1-2x) \cdot (-2) \\ &= 4(1-2x)^3 \cdot (-2) \\ &= -8(1-2x)^3 \end{aligned}$$

By Chain Rule,

$$y = u'(v(x)) \cdot v'(x)$$
$$= u'((1-2x)^4) \cdot (-8(1-2x)^3)$$
$$= \exp[(1-2x)^4] \cdot (-8(1-2x)^3)$$
$$= -8(1-2x)^3 \exp[(1-2x)^4]$$

(d) $y = 3x^2\sqrt{25-x^2}$

Note to find the derivative of this function, we need to apply the product rule with a chain rule on $v(x)$.

Let $u(x) = 3x^2$ $\quad v(x) = \sqrt{25-x^2}$

$$= (25-x^2)^{1/2}$$

Let $a(x) = x^{1/2}$ $\quad b(x) = 25-x^2$

$a'(x) = \frac{1}{2}x^{-1/2}$ $\quad b'(x) = -2x$

By Chain Rule,

$$v'(x) = a'(b(x)) \cdot b'(x)$$
$$= a'(25-x^2) \cdot (-2x)$$
$$= \frac{1}{2}(25-x^2)^{-1/2} \cdot (-2x)$$

$u'(x) = 6x$ $\quad v'(x) = -x(25-x^2)^{-1/2}$

By Product Rule,

$$y = u'(x)v(x) + u(x)v'(x)$$
$$= 6x \cdot (25-x^2)^{1/2} + 3x^2 \cdot (-x(25-x^2)^{-1/2})$$
$$= 6x(25-x^2)^{1/2} - 3x^3(25-x^2)^{-1/2}$$

(e) $y = e^{9x}\cos(7x)$

Note to find the derivative of this function, we need to apply the product rule with chain rule on both $u(x)$ and $v(x)$.

Let $u(x) = e^{9x}$ $\quad v(x) = \cos(7x)$

$u'(x) = 9e^{9x}$ $\quad v'(x) = -\sin(7x) \cdot 7$

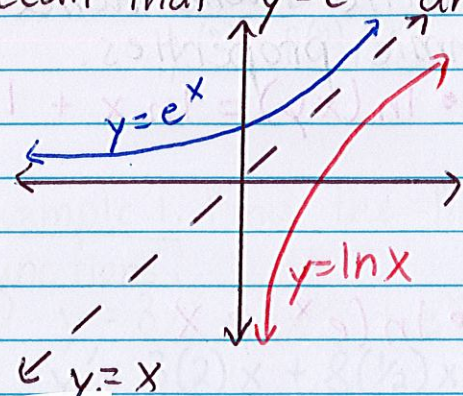
$$= -7\sin(7x)$$

By Product Rule,

$$y = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= (9e^{9x})\cdot(\cos(7x)) + (e^{9x})\cdot(-7\sin(7x))$$
$$= 9e^{9x}\cos(7x) - 7e^{9x}\sin(7x)$$

Derivative of the Natural Logarithmic Functions

Recall that $y = e^x$ and $y = \ln x$ are inverses. More so,



$$\text{Slope of } \ln x = \frac{1}{\text{Slope of } e^x}$$

Hence For $x > 0$,

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Example 2: Find the derivative of the following functions:

ⓐ $y = x\ln x$

Note to find y', we use product rule.

Let $u(x) = x$ $\quad v(x) = \ln x$

$u'(x) = 1$ $\quad v'(x) = \frac{1}{x}$

By Product Rule,

$$y' = u'(x)v(x) + u(x)v'(x)$$
$$= 1\cdot\ln x + x\cdot\frac{1}{x}$$
$$= \ln x + 1$$

ⓑ $y = \ln(3x^2 + x + 1)$

Note to find y', we use Chain Rule.

Let $u(x) = \ln x$ $\quad v(x) = 3x^2 + x + 1$

$u'(x) = 1/x$ $\quad v'(x) = 6x + 1$

By Chain Rule,

$$y' = u'(v(x))\,v'(x)$$
$$= u'(3x^2+x+1)\cdot(6x+1)$$
$$= \frac{1}{3x^2+x+1}\cdot(6x+1)$$
$$= \frac{6x+1}{3x^2+x+1}$$

Sometimes when taking derivatives of logarithmic functions, it is better to rewrite them. Hence we should recall some logarithmic properties.

- $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
- $\ln(xy) = \ln x + \ln y$
- $\ln(x^m) = m\ln x$
- $\ln(1) = 0$
- $\ln(e^x) = x$

<u>Example 3:</u> Find the derivative of $y = \ln\sqrt[3]{\frac{x^2+1}{2x-1}}$

Note this problem can be done with
a Chain Rule → Chain Rule → Quotient Rule
But we can avoid that first rewriting the function.

<u>Rewrite:</u> $y = \ln\left(\frac{x^2+1}{2x-1}\right)^{1/3} = \frac{1}{3}\ln\left(\frac{x^2+1}{2x-1}\right)$

$$= \frac{1}{3}\left(\ln(x^2+1) - \ln(2x-1)\right)$$
$$= \frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(2x-1)$$

Apply Chain Rule to each term, and

$$y' = \frac{1}{3}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{1}{3}\cdot\frac{1}{2x-1}\cdot 2$$
$$= \frac{2x}{3(x^2+1)} - \frac{2}{3(x^2+1)}$$