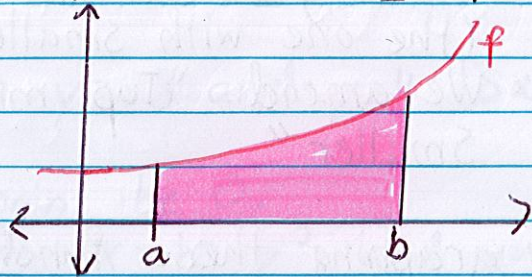


Lesson 13: Area Between Two Curves II

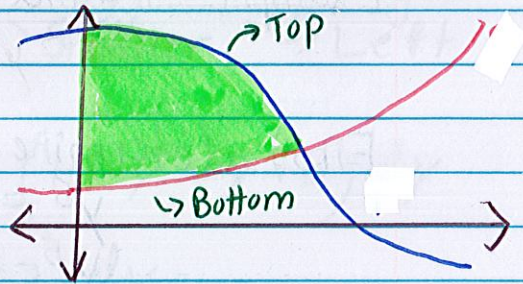
Last time, we recalled from Calculus I that

$$\int_{x=a}^{x=b} f(x) dx \Rightarrow$$



and with that interpretation found a formula for the area between two curves with respect to x , i.e.

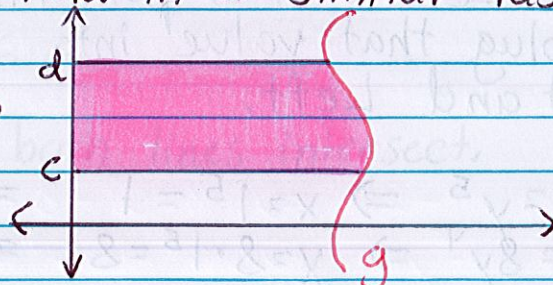
$$\text{Area} = \int_{x=a}^{x=b} (\text{Top} - \text{Bottom}) dx$$



Today's lesson, we will be focussing on the area between two curves with respect to y .

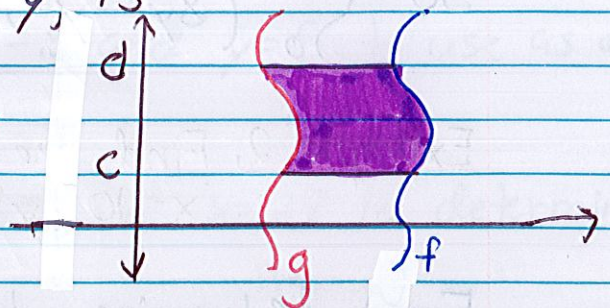
We will define the formula in a similar fashion. So

$$\int_{y=c}^{y=d} g(y) dy \Rightarrow$$



Hence we can say the formula for the area between two curves with respect to y , is

$$\text{Area} = \int_{y=c}^{y=d} (\text{Right} - \text{Left}) dy$$



So what has changed?

- The roles of x and y switch.
- Given two curves which are in terms of y , we want to integrate (the one with larger x -values) minus (the one with smaller x -values).
- We amend "Top minus Bottom" to "Bigger minus Smaller"

Now graphing these functions can be quiet difficult so I'll be introducing a new way of doing this problems,

Example 1: Find the area of the region bounded by $x = y^5$ and $x = 8y^4$

First determine where both lines intersect.

$$\begin{aligned}y^5 &= x = 8y^4 \\y^5 - 8y^4 &= 0 \\y^4(y - 8) &= 0 \\y &= 0, 8\end{aligned}$$

Next choose a # between $y=0$, and $y=8$ to use as a test point.

ex. Let the test point be 1.

Now plug that value into $x = y^5$, and $x = 8y^4$ to determine Right and Left.

$$\begin{aligned}x = y^5 &\Rightarrow x = 1^5 = 1 &\Rightarrow \text{Smaller} &\Rightarrow \text{Left} \\x = 8y^4 &\Rightarrow x = 8 \cdot 1^4 = 8 &\Rightarrow \text{Bigger} &\Rightarrow \text{Right}\end{aligned}$$

$$\text{So } \int_0^8 (8y^4 - y^5) dy = \left(\frac{8y^5}{5} - \frac{y^6}{6} \right) \Big|_0^8 = \frac{131072}{15}$$

Example 2: Find the area of the region bounded by $x = 10 - y^2$, and $x = y - 2$

First determine where both lines intersect.

$$10 - y^2 = y - 2$$

$$0 = y^2 + y - 12$$

$$0 = (y+4)(y-3)$$

$$y = -4, 3$$

Next choose a # between $y = -4$ and $y = 3$ to use as a test point.

ex. Let the test point be 0.

Now plug that value into $x = 10 - y^2$, and $x = y - 2$ to determine Right and Left.

$$x = 10 - y^2 \Rightarrow x = 10 - 0^2 = 10 \Rightarrow \text{Bigger} \Rightarrow \text{Right}$$

$$x = y - 2 \Rightarrow x = 0 - 2 = -2 \Rightarrow \text{Smaller} \Rightarrow \text{Left}$$

$$\text{So } \int_{-4}^3 (10 - y^2 - (y - 2)) dy = \int_{-4}^3 (10 - y^2 - y + 2) dy$$

$$= \int_{-4}^3 (-y^2 - y + 12) dy$$

$$= \left(-\frac{y^3}{3} - \frac{y^2}{2} + 12y \right) \Big|_{-4}^3 = \frac{343}{6}$$

Example 3: Find the area of the region bounded by $x = y^2 + 2$ and $x = 27$

First determine where both lines intersect.

$$y^2 + 2 = 27$$

$$y^2 = 25$$

$$y = \pm 5$$

Next choose a # between $y = -5$ and $y = 5$ to use as a test point.

ex. Let the test point be 0.

Now plug that value into $x = y^2 + 2$ and $x = 27$ to determine Right and Left

$$\begin{aligned} x=y^2+2 &\Rightarrow x=0^2+2=2 &\Rightarrow \text{Smaller} &\Rightarrow \text{Left} \\ x=27 &\Rightarrow x=27 &\Rightarrow \text{Bigger} &\Rightarrow \text{Right} \end{aligned}$$

$$\begin{aligned} \text{So } \int_{-5}^5 (27 - (y^2+2)) dy &= \int_{-5}^5 (27 - y^2 - 2) dy \\ &= \int_{-5}^5 (25 - y^2) dy \\ &= \left(25y - \frac{y^3}{3} \right) \Big|_{-5}^5 = \frac{500}{3} \end{aligned}$$

Example 4: Find the area of the region bounded by $x=y^2-8$ and $x=4-y^2$

First determine where both lines intersect

$$\begin{aligned} y^2-8 &= 4-y^2 \\ 2y^2 &= 12 \\ y^2 &= 6 \\ y &= \pm\sqrt{6} \end{aligned}$$

Next choose a # between $y=-\sqrt{6}$ and $y=\sqrt{6}$ to use as a test point.

ex. Let the test point be 0.

Now plug that value into $x=y^2-8$ and $x=4-y^2$ to determine Right and Left.

$$\begin{aligned} x=y^2-8 &\Rightarrow x=0^2-8=-8 &\Rightarrow \text{Smaller} &\Rightarrow \text{Left} \\ x=4-y^2 &\Rightarrow x=4-0^2=4 &\Rightarrow \text{Bigger} &\Rightarrow \text{Right} \end{aligned}$$

$$\begin{aligned} \text{So } \int_{-\sqrt{6}}^{\sqrt{6}} (4-y^2 - (y^2-8)) dy &= \int_{-\sqrt{6}}^{\sqrt{6}} (4-y^2 - y^2 + 8) dy \\ &= \int_{-\sqrt{6}}^{\sqrt{6}} (12 - 2y^2) dy \\ &= \left(12y - \frac{2y^3}{3} \right) \Big|_{-\sqrt{6}}^{\sqrt{6}} = 16\sqrt{6} \end{aligned}$$