MA 16020 LESSONS 17 + 18: VOLUME BY REVOLUTION – SHELL METHOD (SUPPEMENTAL HOMEWORK)

Solutions

Formulas:

- **Rotating around y-axis:**

$$V = 2\pi \int_a^b x \cdot (Top - Bottom)\, dx$$

- **Rotating around x = #**
  - If $x = \#$ is on the left of your region, then
  $$V = 2\pi \int_a^b (x - \#) \times (Top - Bottom)\; dx$$
  - If $x = \#$ is on the right of your region, then
  $$V = 2\pi \int_a^b (\# - x) \times (Top - Bottom)\; dx$$

- **Rotating around x-axis:**

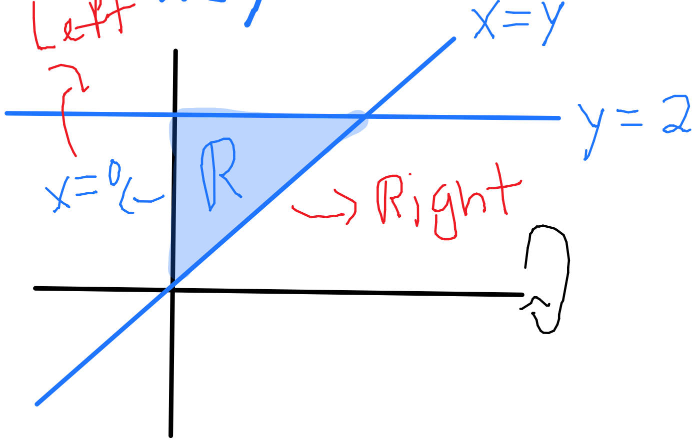
$$V = 2\pi \int_c^d y \cdot (Right - Left)\, dy$$

- **Rotating around y = #**
  - If $y = \#$ is below your region, then
  $$V = 2\pi \int_a^b (y - \#) \times (Right - Left)\; dy$$
  - If $y = \#$ is above your region, then
  $$V = 2\pi \int_a^b (\# - y) \times (Right - Left)\; dy$$

<u>(OPTIONAL HOMEWORK):</u> Set up the integral using the Shell Method that represents the volume of the following solids about the given line:

1. $x = y, \qquad x = 0, \qquad y = 2 \qquad$ **about the x-axis** $\Rightarrow dy$

$x = y$



$$V = 2\pi \int y\,(Right - Left)\,dy$$

$$V = 2\pi \int_0^2 y\,(y - 0)\,dy$$

$$\boxed{V = 2\pi \int_0^2 y^2\,dy}$$

2. $x = 2y - y^2, \quad x = 0$ about the x-axis $\Rightarrow dy$

Bounds: $0 = 2y - y^2$

$0 = y(2-y)$

$y = 0, 2$

$$\boxed{V = 2\pi \int_0^2 y(2y - y^2)\,dy}$$

3. $y = \sqrt{x}, \quad y = -x$ about the y-axis $\Rightarrow dx$

Bounds: $\sqrt{x} = x$

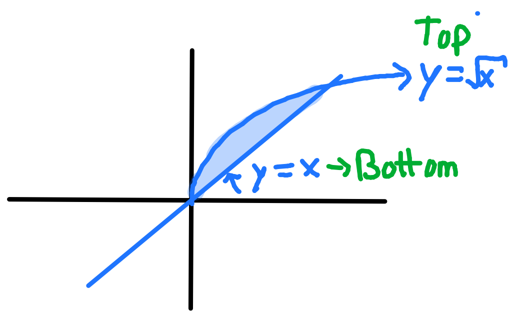
$(\sqrt{x})^2 = x^2$

$x = x^2$

$x - x^2 = 0$

$x(1-x) = 0$

$x = 0, 1$



$$\boxed{V = 2\pi \int_0^1 x(\sqrt{x} - x)\,dx}$$

4. $y = 2 - x^2, \quad y = x^2$ about the y-axis $\Rightarrow dx$

Bounds: $2 - x^2 = x^2$

$2 = 2x^2$

$1 = x^2$

$x = \pm 1$

Test Pt: $x = 0$

$y = 2 - x^2 \rightarrow y = 2 \rightarrow$ Top

$y = x^2 \rightarrow y = 0 \rightarrow$ Bottom

$$V = 2\pi \int_{-1}^{1} x(2 - x^2 - x^2)\,dx$$

$$\boxed{V = 2\pi \int_{-1}^{1} x(2 - 2x^2)\,dx}$$

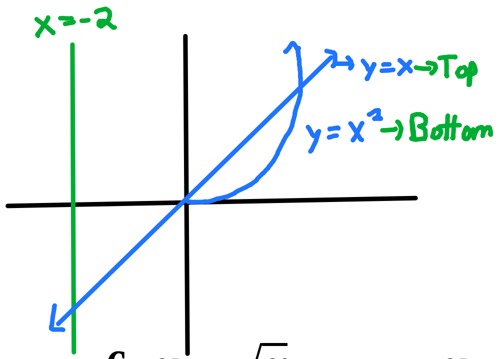
5. $y = x, \quad y = x^2 \quad$ about $x = -2$ → dx

Bounds: $x = x^2$

$x - x^2 = 0$

$x(1-x) = 0$

$x = 0, 1$

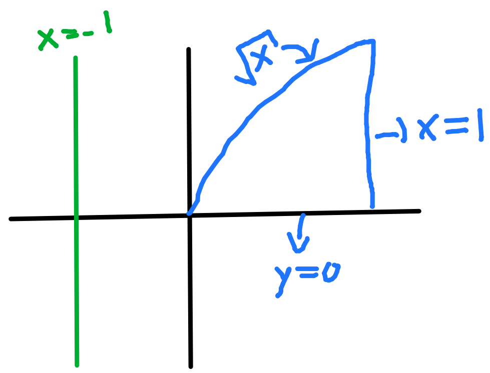


Since $x=-2$ is smaller than the bounds,

$$V = 2\pi \int_0^1 \underbrace{(x-(-2))}\left[x - x^2\right]dx$$

$$\boxed{V = 2\pi \int_0^1 (x+2)(x-x^2)\,dx}$$

6. $y = \sqrt{x}, \quad y = 0, \quad x = 1 \quad$ about $x = -1$ → dx



$$V = 2\pi \int_0^1 (x-(-1))\sqrt{x}\,dx$$

$$\boxed{V = 2\pi \int_0^1 (x+1)\sqrt{x}\,dx}$$

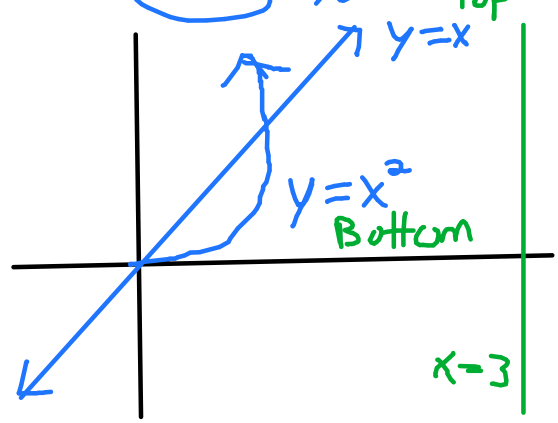
7. $y = x, \quad y = x^2 \quad$ about $x = 3$ → dx

Bounds: $x = x^2$

$x - x^2 = 0$

$x(1-x) = 0$

$x = 0, 1$



Since $x=3$ is larger than the bounds,

$$V = 2\pi \int_0^1 (3-x)(x-x^2)\,dx$$

8. $y = 4x - x^2$, $y = 3$ about $x = 1$ $\Rightarrow dx$

Bounds: $4x - x^2 = 3$

$0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

$x = 1, 3$

Since $x=1$ is equal to the bottom bound

$$\boxed{V = 2\pi \int_1^3 (x-1)(4x - x^2 - 3)\,dx}$$

Test Pt: $x = 2$

$y = 4x - x^2 \rightarrow y = 4 \rightarrow$ Top

$y = 3 \rightarrow y = 3 \rightarrow$ Bottom

9. $x = 2y - y^2$, $x = 1$, $y = 0$ about $y = -1$ $\Rightarrow dy$

Bounds: $2y - y^2 = 1$

$0 = y^2 - 2y + 1$

$0 = (y-1)^2$

$0 = y - 1$

$y = 1$

The other bound is given $y = 0$

Test Pt: $y = \frac{1}{2}$

$x = 2y - y^2 \rightarrow x = \frac{3}{4} \rightarrow$ Left

$x = 1 \rightarrow x = 1 \rightarrow$ Right

Since $y=-1$ is smaller than bounds,

$$V = 2\pi \int_0^1 (y-(-1))(1-(2y-y^2))\,dy$$

$$\boxed{V = 2\pi \int_0^1 (y+1)(1 - 2y + y^2)\,dy}$$

10. $x = y^2 + 1$, $x = 2$ about $y = -2$ $\Rightarrow dy$

Bounds: $y^2 + 1 = 2$

$y^2 = 1$

$y = \pm 1$

Test Pt: $y = 0$

$x = y^2 + 1 \rightarrow x = 1 \rightarrow$ Left

$x = 2 \rightarrow x = 2 \rightarrow$ Right

Since $y = -2$ is smaller than the bounds

$$V = 2\pi \int_{-1}^1 (y-(-2))(2-(y^2+1))\,dy$$

$$\boxed{V = 2\pi \int_{-1}^1 (y+2)(2-(y^2+1))\,dy}$$

11. $x = 4y^2 - y^3, \quad x = 0$ **about** $y = 6 \Rightarrow dy$

Bounds: $4y^2 - y^3 = 0$

$y^2(4-y) = 0$

$y = 0, 4$

Since $y=6$ is larger than the bounds,

$$\boxed{V = 2\pi \int_0^4 (6-y)(4y^2-y^3)\,dy}$$

12. $x = (y-3)^2, \quad x = 4$ **about** $y = 1 \Rightarrow dy$

Bounds: $(y-3)^2 = 4$

$y - 3 = \pm 2$

$y = 3 \pm 2$

$y = 1, 5$

Test Pt: $y = 2$

$x = (y-3)^2 \rightarrow x = 1 \rightarrow$ left

$x = 4 \rightarrow x = 4 \rightarrow$ right

Since $y=1$ is equal to the lower bound,

$$\boxed{V = 2\pi \int_1^5 (y-1)(4-(y-3)^2)\,dy}$$