MA 16020: Lesson 17
Volume By Revolution
Shell Method
Pt 1: Rotation around the x- or y-axis

By Alexandra Cuadra

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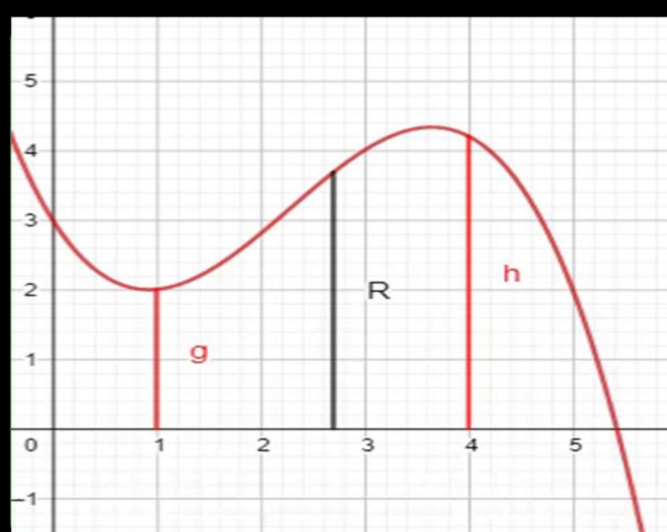
So far…

- We have learned how to find the volume of a solid of revolution by integrating
  - In the same way, we calculate the area under a curve
    - Running a line segment of varying length across the region, and adding them up



https://www.geogebra.org/m/tgceabb2#material/tnnhu7gz

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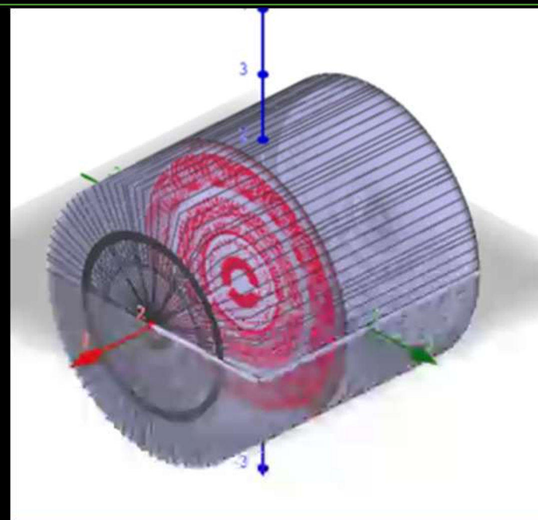
In other words,

- We learned to find the volume of a solid of revolution by
  - Running some area across a shape and add them up.
  - Like in the case of the cylinder shown on the right.



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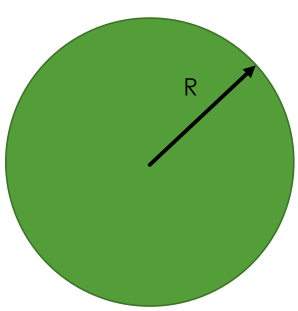
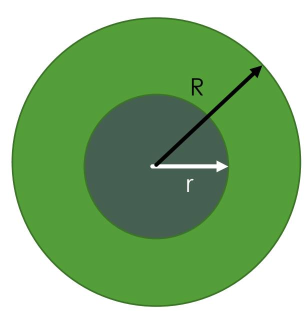
But what were those "shapes"? ANSWER: CROSS-SECTIONS

Sometimes it was a disk



Whose area is πR^2

Sometimes it was a washer



Whose area is $\pi(R^2 - r^2)$

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In Today's Lecture, we will be covering the case, when neither method (Disk nor Washer) is easy.

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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

https://www.geogebra.org/m/jqfyndpu

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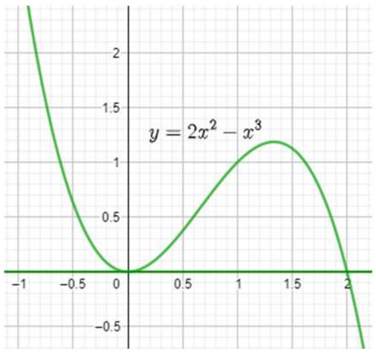
Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

Draw the region.



https://www.geogebra.org/m/jqfyndpu

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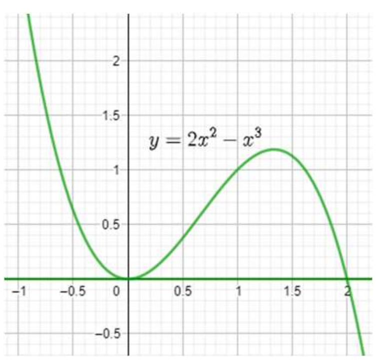
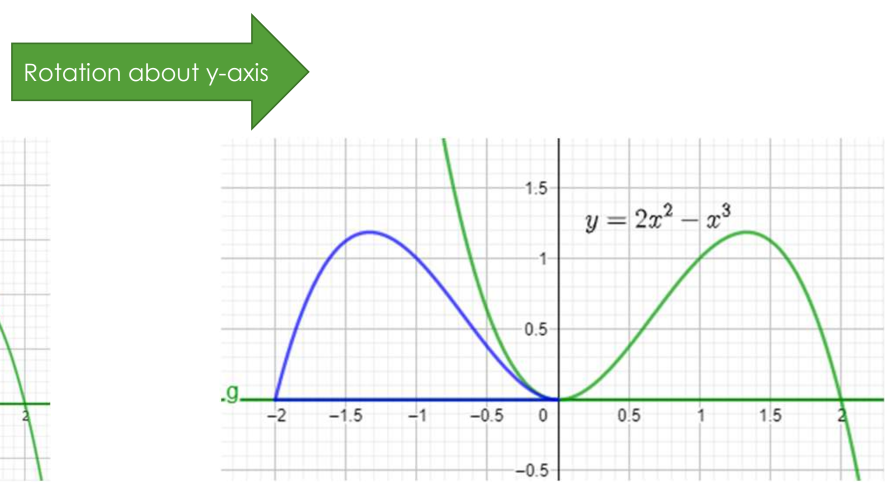
Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.





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Example 1: Find the volume obtained by revolving the region bounded by the curves

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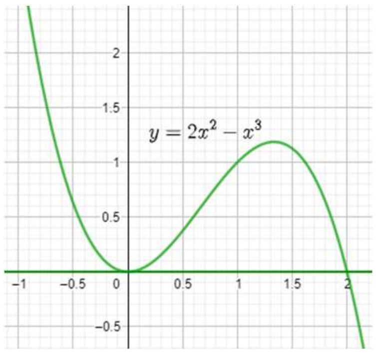
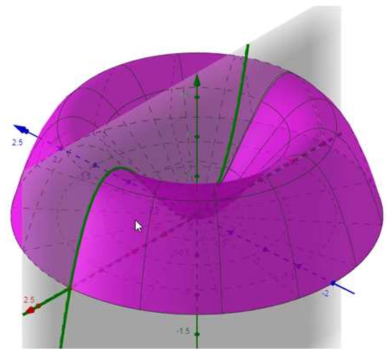
About the y-axis.



WASHER ???

Furthermore, 3-D



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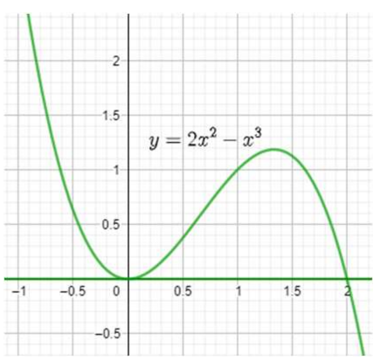
Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \qquad y = 0$$

About the y-axis.

Technically, yes. It is a Washer Problem.
But there are two issues:



1. Given we are revolving around y-axis, we want to solve our equations for x.

   i.e. Solve $y = 2x^2 - x^3$ for x.

   But that is easier said than done.

2. For washer problems, we need two equations for each radius.

   Here we have both radius depend on the same function.

https://www.geogebra.org/m/jqfyndpu

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So how can I do this kind of problem without giving myself a headache?

ANSWER: SHELL METHOD

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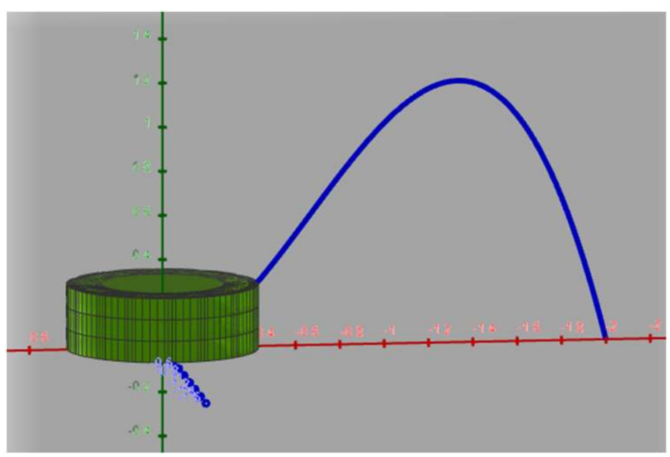
What's a (Cylindrical) Shell?

- Before we would find the volume by taking cuts perpendicular to the axis,
  - **In Shells, we take cuts parallel to our axis (as shown in the image in the right)**
- The reason is USEFUL is that
  - For this problem, we no longer have to solve for x in terms of y.
- **What's the formula of that shell?**



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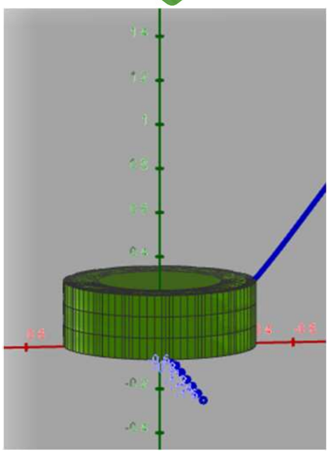
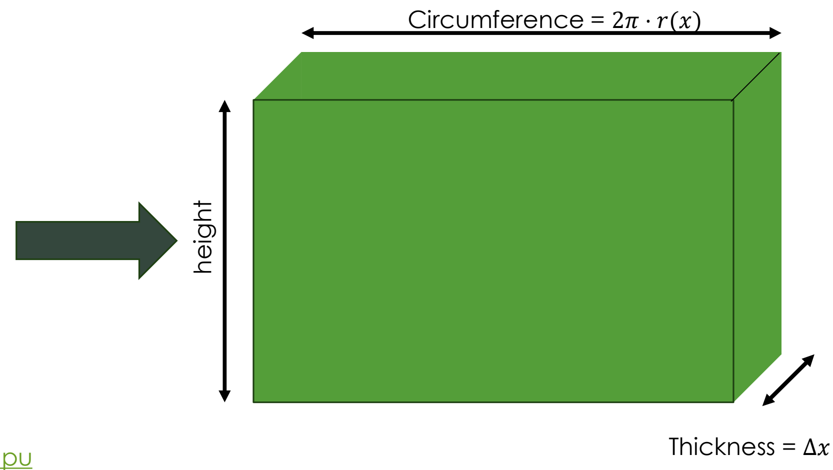
Geometry Time: Let's Flatten the Shell …





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Geometry Time: Let's Flatten the Shell …

- So the volume of the **green image** is

  V = circumference × height × thickness

  $V = 2\pi \cdot \mathrm{r}(x) \cdot h \cdot \Delta x$

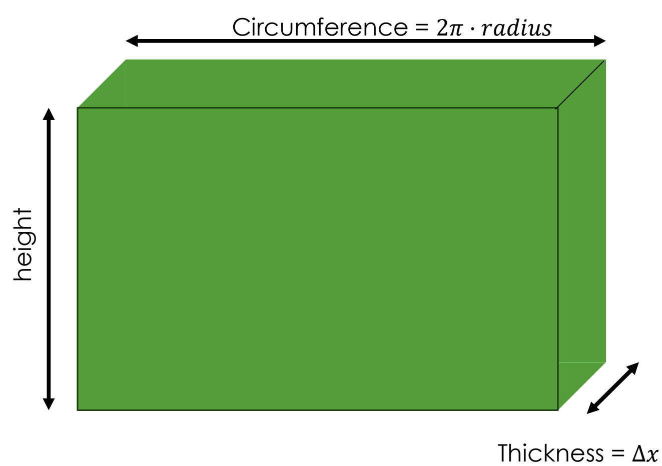
  where $r(x)$ is the radius.

- The height is determined if you have one or two functions.
  - i.e. Top - Bottom or Right – Left

- So, in the dx case,

  $V = 2\pi \cdot r(x)\ (Top - Bottom)\ dx$ over $[a, b]$.

  i.e. $V = 2\pi \int_a^b r(x) \cdot (Top - Bottom)\, dx$



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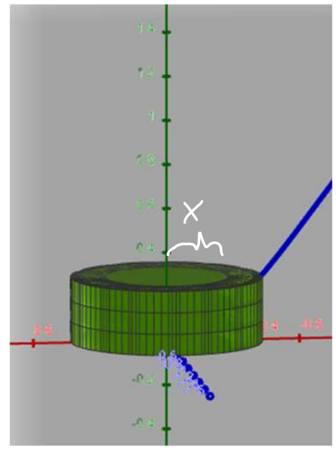
But what is the radius, $r(x)$?

- To find $r(x)$, we need to find the distance of the shell from the axis of rotation
- So, in the dx case with rotation around the y-axis,
  - The shell is x units away from the y-axis. So,

$$r(x) = x$$

- This yields the formula:

$$V = 2\pi \int_a^b x \cdot (Top - Bottom)\, dx$$



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One thing about Shell Method Formulas

Since we are just cutting out parallel to the axis, we choose dx or dy in the following way:

- **Rotating around y-axis**
  ⇒ " dx " problem

$$V = 2\pi \int_a^b x \cdot (Top - Bottom)\, dx$$

- **Rotating around x-axis**
  ⇒ " dy " problem

$$V = 2\pi \int_c^d y \cdot (Right - Left)\, dy$$

If you need more of an explanation of where the Shell Method comes look at the hidden slides.

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GEOMETRY: Finding The Volume of A Hollow Cylinder

- To find the volume of this hollow cylinder, we used the same idea when washers were first introduced.

$$V_{total} = V_{outer} - V_{inner}$$

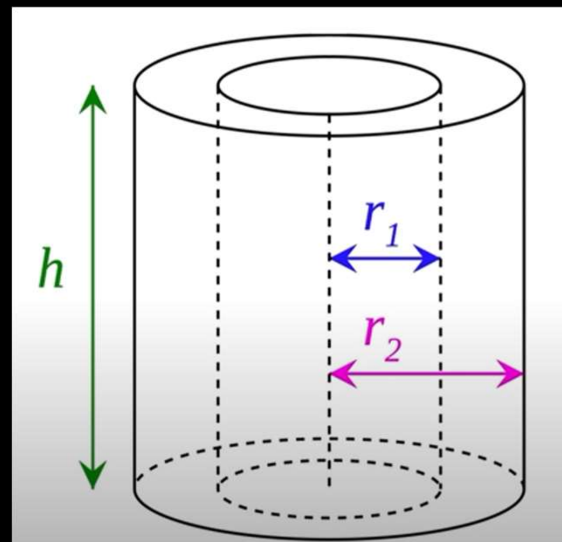
- Remember the volume of a cylinder is $\pi r^2 h$. So

$V_{outer} = \pi(r_2)^2 h$ and $V_{inner} = \pi(r_1)^2 h$

- Hence $V_{total} = \pi r_2^2 h - \pi r_1^2 h$

$$= \pi h(r_2^2 - r_1^2)$$

$$= \pi h(r_2 - r_1)(r_2 + r_1)$$



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GEOMETRY: Finding The Volume of A Hollow Cylinder

So let's be clever

- Let's take the sum $r_2 + r_1$ and express it as an average.

i.e. $(r_1 + r_2)/2$

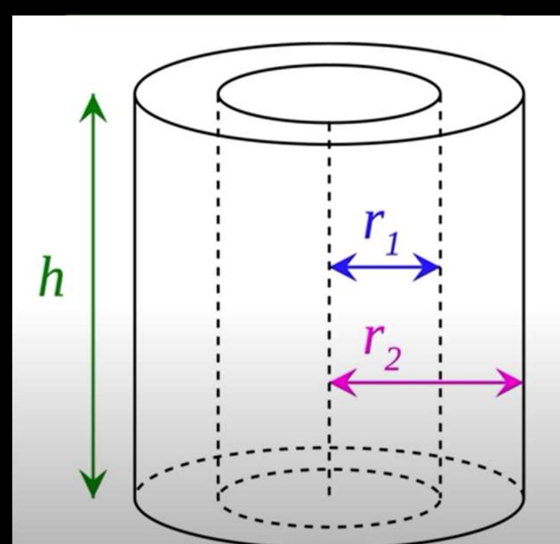
- To do that multiple the equation below by 2/2.

$$V_{total} = \pi h(r_2 - r_1)(r_2 + r_1)$$

$$= 2\pi h(r_2 - r_1)\left(\frac{r_2 + r_1}{2}\right)$$

- Since we have the average radius in our equation, we can now call $r = \frac{r_1 + r_2}{2}$.

$$V_{total} = 2\pi h(r_2 - r_1)r$$



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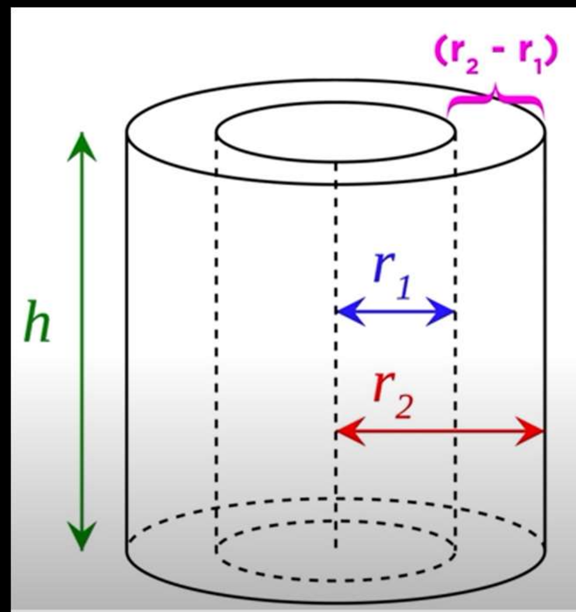
GEOMETRY: Finding The Volume of A Hollow Cylinder

- Note that the difference of the radii gives us the thickness of the cylinder.
  - Let Δr be that difference

$$\Delta r = r_2 - r_1$$

- Hence we can say that the volume of the hollow cylinder is

$$V_{total} = 2\pi rh \cdot \Delta r$$



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GEOMETRY: Finding The Volume of A Hollow Cylinder

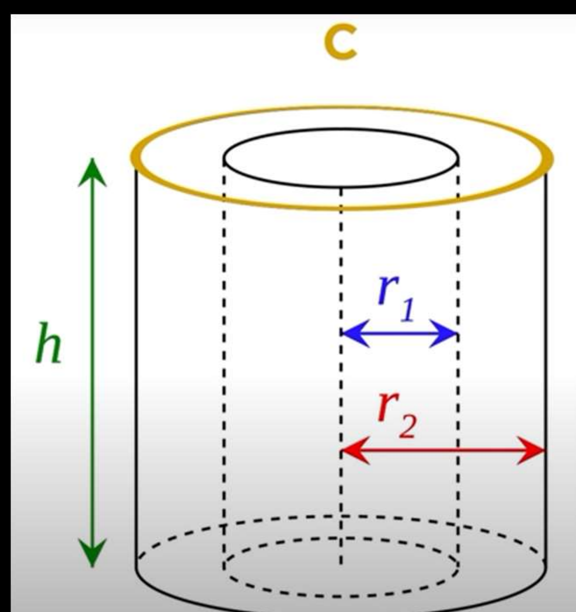
- One way to remember this

$$V_{total} = 2\pi rh \cdot \Delta r$$

is to see that $2\pi r$ is the same as the circumference, C, (as shown in the image) of the cylinder.

- So this is just the

circumference × height × thickness.



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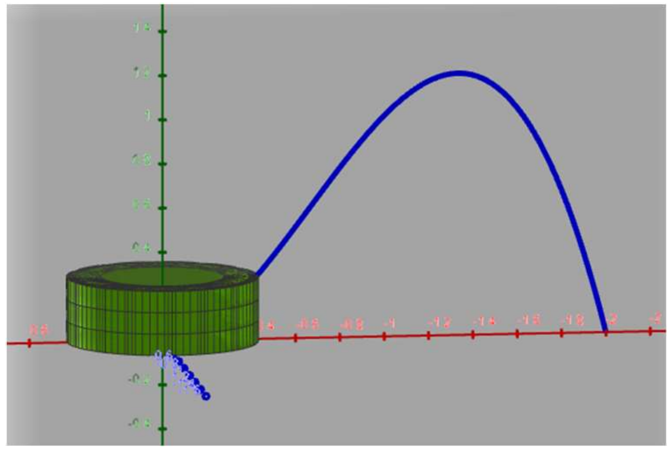
So how does this help us answer Example 1?

- The reason is USEFUL is that
  - For this problem, we no longer have to solve for x in terms of y.
- If we picture one possible shell, it will have a
  - $Radius = x$
  - $height = f(x)$
  - $circumference = 2\pi x$
- As this shell spans the volume, we then have

$$V = \int_a^b 2\pi x \cdot f(x)dx$$



https://www.geogebra.org/m/jqfyndpu

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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis. ⇒ dx problem

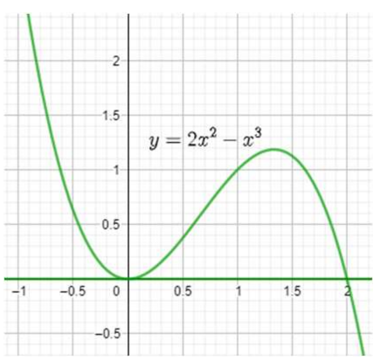
Find the bounds by setting the eqns equal

$2x^2 - x^3 = 0$

$x^2(2-x) = 0$

$x = 0, 2$

So $V = 2\pi \int_0^2 x(2x^2 - x^3)dx$



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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

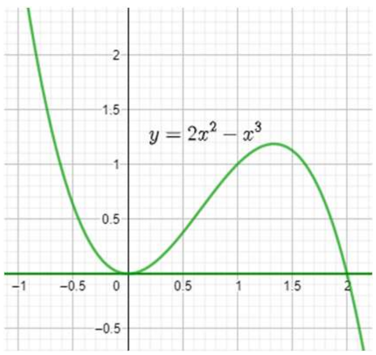
About the y-axis.

$$V = 2\pi \int_0^2 x(2x^2 - x^3)\,dx$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$$

$$= 2\pi \left(\frac{2x^4}{4} - \frac{x^5}{5} \right) \Big]_0^2$$

$$= \frac{16\pi}{5}$$



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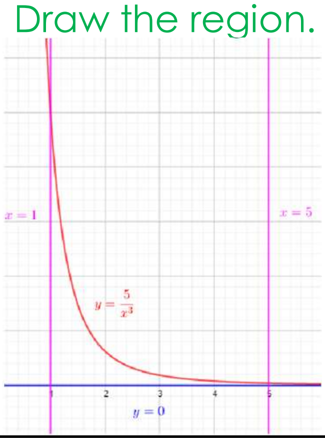
Example 2: Find the volume obtained by revolving the region bounded by the curves

$$y = \frac{5}{x^3}, \quad y = 0, \quad x = 1 \quad \text{and} \quad x = 5$$

About the y-axis. ⇒ dx problem

Draw the region.



$$V = 2\pi \int_1^5 x\left(\frac{5}{x^3}\right)dx$$

$$= 2\pi \int_1^5 \left(\frac{5}{x^2}\right)dx$$

$$= 10\pi \int_1^5 x^{-2}\,dx$$

$$= 10\pi \left(-x^{-1}\right)\Big]_1^5$$

$$= -10\pi \left(\frac{1}{x}\right)\Big]_1^5$$

$$= 8\pi$$

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Example 3: Find the volume obtained by revolving the region bounded by the curves

$$y = \sqrt{256x}, \quad \text{and} \quad y = 2x^2$$

About the y-axis. $\Rightarrow dx$ problem

Draw the region.

First find the bounds

$$\sqrt{256x} = 2x^2$$
$$256x = 4x^4$$
$$0 = 4x^4 - 256x$$
$$0 = 4x(x^3 - 64)$$
$$x = 0, 4$$

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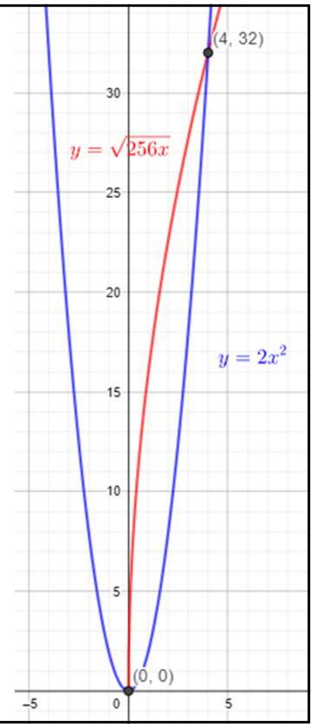

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Example 3: Find the volume obtained by revolving the region bounded by the curves

$$y = \sqrt{256x}, \quad \text{and} \quad y = 2x^2$$

About the y-axis. $\Rightarrow dx$ problem

Draw the region.

From the graph we can figure out the Top and Bottom, and get

$$V = 2\pi \int_0^4 x\left(\sqrt{256x} - 2x^2\right) dx$$
$$= 2\pi \int_0^4 x\left(\sqrt{256}\sqrt{x} - 2x^2\right) dx$$

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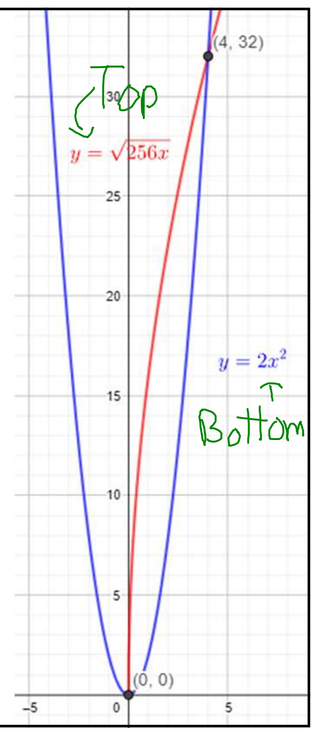

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Example 3: Find the volume obtained by revolving the region bounded by the curves

$$y = \sqrt{256x}, \quad \text{and} \quad y = 2x^2$$

About the y-axis.

Draw the region.

$$V = 2\pi \int_0^4 x(16x - 2x^2)\,dx$$

$$= 2\pi \int_0^4 [16x^2 - 2x^3]\,dx$$

$$= 2\pi \left(\frac{16x^3}{3} - \frac{2x^4}{4} \right) \Big]_0^4$$

$$= \frac{1280\pi}{3}$$

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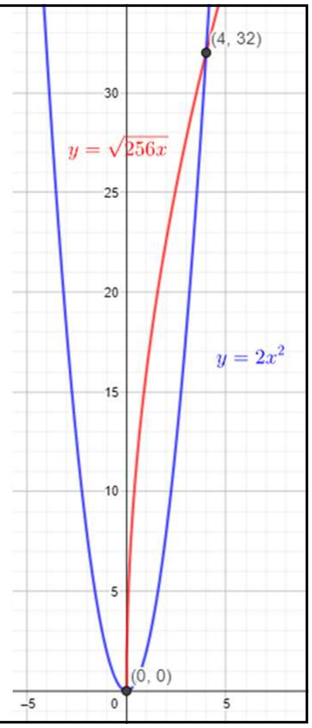

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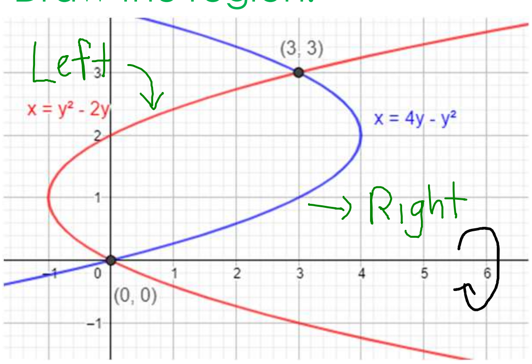
Example 4: Find the volume obtained by revolving the region bounded by the curves

$$x = y^2 - 2y \quad \text{and} \quad x = 4y - y^2$$

About the x-axis. ⟹ dy problem w/ Right − Left

Draw the region.



Find the bounds by setting the eqns equal

$$y^2 - 2y = 4y - y^2$$

$$2y^2 - 6y = 0$$

$$2y(y-3) = 0$$

$$y = 0, 3$$

So $V = 2\pi \int_0^3 x[y^2 - 2y - (4y - y^2)]\,dy$

https://www.geogebra.org/m/f3wrypfh#material/qrar4br5

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Example 4: Find the volume obtained by revolving the region bounded by the curves

$$x = y^2 - 2y \quad \text{and} \quad x = 4y - y^2$$

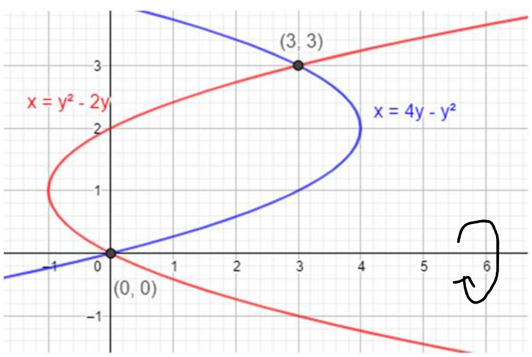
About the x-axis.

$V = 2\pi \int_0^3 y(2y^2 - 6y)\,dy$

$= 2\pi \int_0^3 (2y^3 - 6y^2)\,dy$

$= 2\pi \left(\frac{2y^4}{4} - \frac{6y^3}{3}\right)\Big]_0^3$

$= 27\pi$



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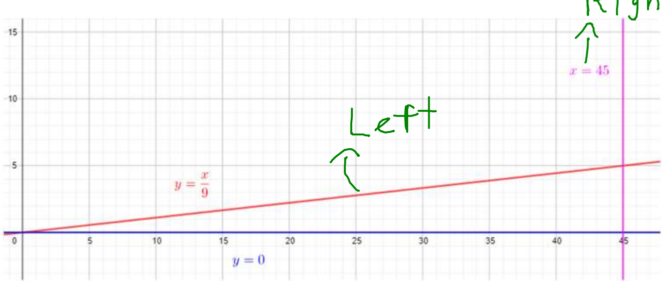
Example 5: Find the volume obtained by revolving the region bounded by the curves

$$y = \frac{x}{9}, \qquad x = 45, \qquad \text{and} \qquad y = 0$$

About the x-axis.

dy problem w/ Right-Left



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Note we can easily find which graph is Right or Left But we need both eqns to be x = (something) So

$y = \frac{x}{9} \iff x = 9y$

So $V = 2\pi \int_0^5 y(45 - 9y)\,dy$

Remember that the bounds are y-bounds So y = 0 can easily be seen on the graph. We find y = 5 by plugging x = 45 into $y = \frac{x}{9}$

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Example 5: Find the volume obtained by revolving the region bounded by the curves

$$y = \frac{x}{9}, \qquad x = 45, \quad \text{and} \quad y = 0$$

About the x-axis.

$V = 2\pi \int_0^5 y(45 - 9y)\,dy$

$= 2\pi \int_0^5 (45y - 9y^2)\,dy$

$= 2\pi \left(\frac{45y^2}{2} - \frac{9y^3}{3} \right) \Big]_0^5$

$= 375\pi$

https://www.geogebra.org/m/f3wrypfh#material/gvkt6rya

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Example 6: Consider the region bounded by:

$$y = 4x\,, \quad y = 0\,, \quad \text{and} \quad x = 10$$

Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

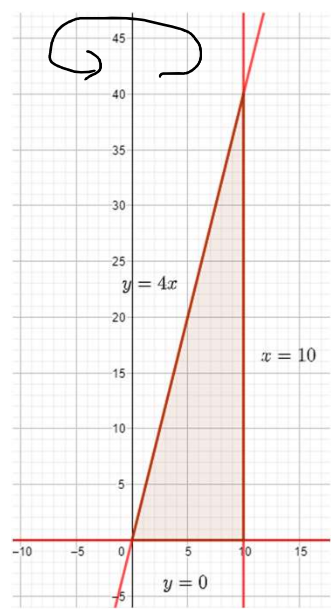
ONLY SET-UP

A) Disk/Washer Method

Draw the region.

y-axis + Disk Washer

$\Rightarrow$ dy problem



https://www.geogebra.org/m/f3wrypfh#material/cb7tfq8n

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Example 6: Consider the region bounded by:

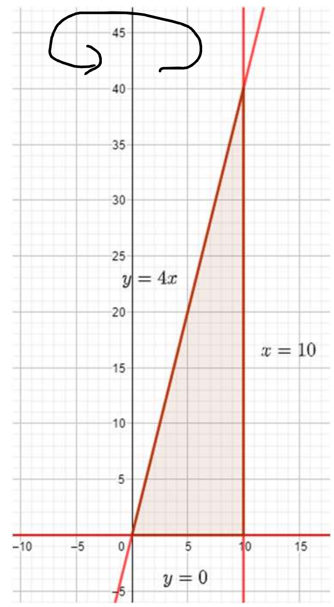
$y = 4x\,, \quad y = 0\,,$ and $x = 10$

Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

A) Disk/Washer Method

Right ⇒ ⇒ x = 10
Left ⇒ y = 4x ⇒ x = y/4

$V = \pi \int_0^{40} \left[(10)^2 - \left(\frac{y}{4}\right)^2\right] dy$



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Example 6: Consider the region bounded by:

$y = 4x\,, \quad y = 0\,,$ and $x = 10$

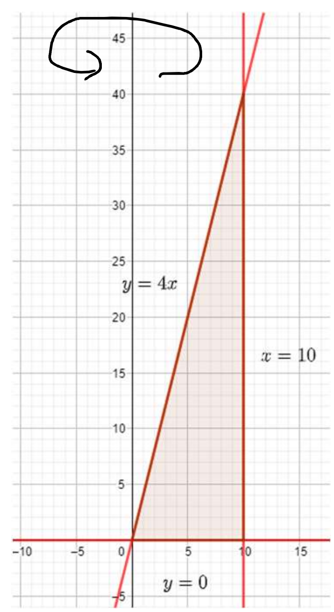
Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

ONLY SET-UP

B) Shell Method

Draw the region.

y-axis + Shell
⇒ dx problem



https://www.geogebra.org/m/f3wrypfh#material/cb7tfq8n

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Example 6: Consider the region bounded by:

$y = 4x\,, \quad y = 0\,,$ and $x = 10$

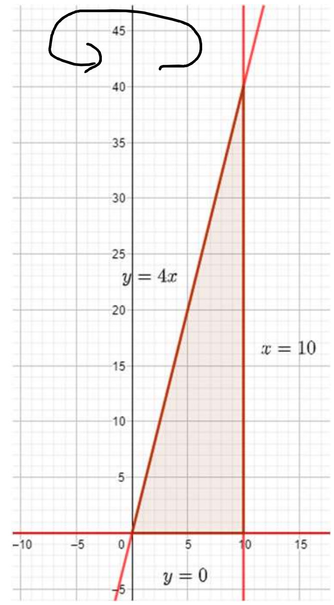
Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

B) Shell Method

Top $\Rightarrow y = 4x$

Bottom $\Rightarrow y = 0$

$V = 2\pi \int_0^{10} x(4x-0)\,dx$



https://www.geogebra.org/m/f3wrypfh#material/cb7tfq8n

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Example 6: Consider the region bounded by:

$y = 4x\,, \quad y = 0\,,$ and $x = 10$

Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

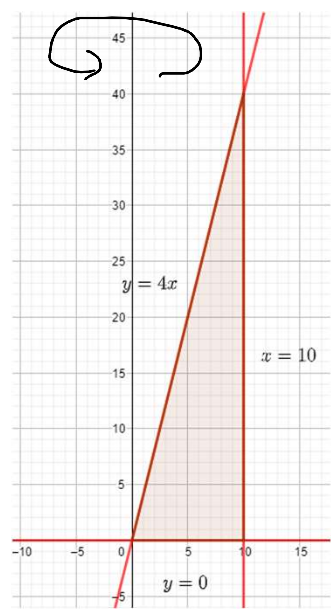
Interesting Question: Which integral is easier to compute?

A) Disk/Washer Method

$$V = \pi \int_0^{40} (10)^2 - \left(\frac{y}{4}\right)^2 \, dy$$

B) Shell Method

$$V = 2\pi \int_0^{10} x(4x)\, dx$$



https://www.geogebra.org/m/f3wrypfh#material/cb7tfq8n

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When do we apply Disk Method or Washer Method or Shell Method?

- When the region **"hugs"** the axis of rotation
  ⇒ **Disk Method**

- When there is a **"gap"** between the region and axis of rotation
  ⇒ **Washer Method**

- But if you find **solving for x or y**, in either method, **is hard**
  ⇒ **Shell Method**

		Axis of Rotation	
		x-axis	y-axis
Method	Disk/Washer	dx	dy
	Shells	dy	dx

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Formulas from Lessons 14 and 15 and 17 Rotation around x-axis or y-axis

For rotation around x-axis:

- Disk Method:

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

- Washer Method:

$$V = \pi \int_a^b (R^2 - r^2) \, dx$$

- Shell Method:

$$V = 2\pi \int_c^d y \cdot (Right - Left) \, dy$$

For rotation around y-axis:

- Disk Method:

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

- Washer Method:

$$V = \pi \int_c^d (R^2 - r^2) \, dy$$

- Shell Method:

$$V = 2\pi \int_a^b x \cdot (Top - Bottom) \, dx$$

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GeoGebra Link for Lesson 17

- https://www.geogebra.org/m/f3wrypfh
- Note click on the play buttons on the left-most screen and the animation will play/pause.

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