Lesson 20: Separation of Variables II

Today's lecture we will be doing more of the same, plus some application problems.

Example 1: Find the general solution of the differential eqn:

ⓐ $t^2 \frac{dy}{dt} - y = 0$

Rewrite: $t^2 \frac{dy}{dt} = y$

$$\frac{dy}{y} = \frac{dt}{t^2}$$

$$\frac{dy}{y} = t^{-2}dt$$

$$\int \frac{dy}{y} = \int t^{-2}dt$$

$$\ln|y| = -t^{-1} + c$$

$$\ln|y| = \frac{-1}{t} + c$$

$$|y| = \exp\left[\frac{-1}{t} + c\right]$$

$$\pm y = e^c \exp\left[\frac{-1}{t}\right]$$

$$y = \underbrace{\pm e^c}_{\text{All a constant}} \exp\left[\frac{-1}{t}\right]$$

$$y = C\exp\left[\frac{-1}{t}\right]$$

ⓑ $-x^3 y + y' = 2x^3$

Rewrite: $y' = 2x^3 + x^3 y$

$$\frac{dy}{dx} = x^3(2+y)$$

$$\frac{dy}{2+y} = x^3 dx$$

$$\int \frac{dy}{2+y} = \int x^3\,dx$$

$$\ln|2+y| = \frac{x^4}{4} + C$$

$$|2+y| = \exp\left[\frac{x^4}{4} + C\right]$$

$$\pm(2+y) = e^C \exp\left[\frac{x^4}{4}\right]$$

$$2+y = \underbrace{\pm e^C}_{\text{All a constant}} \exp\left[\frac{x^4}{4}\right]$$

$$2+y = C\exp\left[\frac{x^4}{4}\right]$$

$$y = C\exp\left[\frac{x^4}{4}\right] - 2$$

ⓒ $3x^2y' = y' + 6x$

Rewrite: $3x^2y' - y' = 6x$

$$(3x^2-1)y' = 6x$$

$$(3x^2-1)\frac{dy}{dx} = 6x$$

$$dy = \frac{6x}{3x^2-1}\,dx$$

$$\int dy = \int \underbrace{\frac{6x}{3x^2-1}}_{\substack{u=3x^2-1 \\ du=6x\,dx}}\,dx$$

$$\int dy = \int \frac{du}{u}$$

$$y = \ln|u|$$

$$y = \ln|3x^2-1| + C$$

Example 2: A wet towel hung on a clothesline to dry outside loses moisture at a rate proportional to its moisture content. After 1 hour, the towel has lost 32% of its original moisture content. After how long will the towel have lost 74% of its moisture content?

Let $M(t)$ = % moisture in t hrs, and

proportional $\Rightarrow M' = kM \Rightarrow M = Ce^{kt}$

and $M(0) = 1$ (b/c 100%) $M(1) = 1 - 0.32 = 0.68$

When $M(0) = 1$,
$1 = Ce^0 = C$
So $M = 1 \cdot e^{kt} = e^{kt}$

When $M(1) = 0.68$
$0.68 = e^k$
$\ln(0.68) = k$
So $M = \exp[t \ln(0.68)]$

Solve $M(t) = 0.26$ for t. b/c 74% lost $\Rightarrow$ 26% moisture

$0.26 = \exp[t \ln(0.68)]$

$\ln(0.26) = t \ln(0.68)$

$t = \dfrac{\ln(0.26)}{\ln(0.68)} \approx 3.493$

Example 3: In a particular chemical reaction, a substance is converted into a second substance at a rate proportional to the square of the amount of the first substance present at any time, t. Initially, 50 grams of the first substance was present, and 1 hour later only 14 grams of the first substance remained. What is the amount of the first substance remaining after 7 hours?

Set-Up: $\frac{da}{dt} = a^2 k$; $a(0) = 50$; $a(1) = 14$

Solve: $\frac{da}{a^2} = k\,dt$

$$\int a^{-2}\,da = \int k\,dt$$

$$-a^{-1} = kt + C$$

$$-\frac{1}{a} = kt + C$$

$$\frac{1}{a} = -kt \underbrace{- C}_{\text{All a constant}}$$

$$\frac{1}{a} = -kt + C$$

$$a = \frac{1}{-kt + C}$$

When $a(0) = 50$,

$$50 = a(0) = \frac{1}{C}$$

$$C = 1/50$$

So $a = \frac{1}{1/50 - kt}$

$$= \frac{50}{1 - 50kt}$$

When $a(1) = 14$,

$$14 = a(1) = \frac{50}{1 - 50k}$$

$$14(1 - 50k) = 50$$

$$1 - 50k = \frac{50}{14} = \frac{25}{7}$$

$$-50k = \frac{25}{7} - 1 = \frac{18}{7}$$

$$k = \frac{-1}{50} \cdot \frac{18}{7}$$

$$= \frac{-18}{350}$$

So $a = \frac{50}{1 - 50\left(\frac{-18}{350}\right)t}$

$$= \frac{350}{7 + 18t}$$

Hence $a(7) \approx 2.6316$ grams

Example 4: The rate of change in the number of miles of road cleared per hour by a snowplow with respect to the depth of the snow is inversely proportional to the depth of the snow. Given that 24 miles per hour are cleared when the depth of the snow is 2.1 inches and 13 miles per hour are cleared when the depth of the snow is 8 inches, then how many miles of road will be cleared each hour when the depth of the snow is 13 inches?

Inversely Proportional $\Rightarrow y' = \frac{k}{x} \Rightarrow \frac{dy}{dx} = \frac{k}{x}$

y-miles and x-depth of snow $\Rightarrow (2.1, 24)$ and $(8, 13)$

WANT: $y(13)$.

① First solve $\frac{dy}{dx} = \frac{k}{x}$

Rewrite: $dy = \frac{k}{x}\,dx$

$$\int dy = \int \frac{k}{x}\,dx$$

$$y = k\ln|x| + c$$

② When $(2.1, 24)$

$$24 = k\ln(2.1) + c \quad (a)$$

When $(8, 13)$

$$13 = k\ln(8) + c \quad (b)$$

③ Thus we have the following system of eqns to solve

$$\begin{cases} 24 = k\ln(2.1) + c & (a) \\ 13 = k\ln(8) + c & (b) \end{cases}$$

③ (continued)

Subtract (b) from (a), i.e.

$$\begin{array}{r} 24 = k\ln(2.1) + c \\ -(13 = k\ln(8) + c) \\ \hline 11 = k\ln(2.1) - k\ln(8) \end{array}$$

$$11 = k[\ln(2.1) - \ln(8)]$$

$$k = \frac{11}{\ln(2.1) - \ln(8)}$$

Plug k into (b).

$$13 = k\ln(8) + c$$

$$13 = \frac{11}{\ln(2.1) - \ln(8)}\ln(8) + c$$

$$13 - \frac{11\ln(8)}{\ln(2.1) - \ln(8)} = c$$

④ Plug k and c found in ③ into the eqn found in ①.

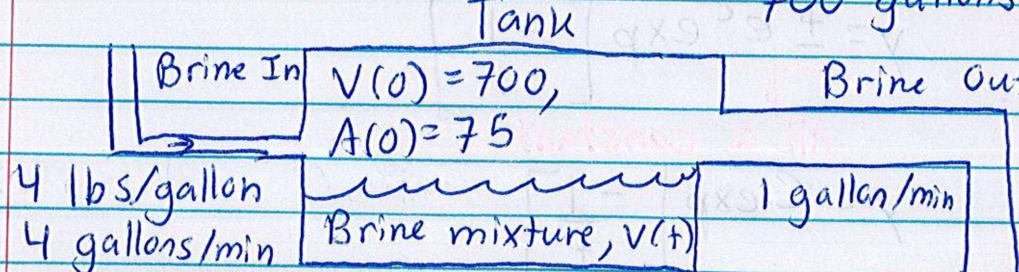
$$y = \frac{11}{\ln(2.1) - \ln(8)}\ln|x| + 13 - \frac{11\ln(8)}{\ln(2.1) - \ln(8)}$$

⑤ Remember we want $y(13)$.

$$y(13) = \frac{11}{\ln(2.1) - \ln(8)} \ln(13) + 13 - \frac{11\ln(8)}{\ln(2.1) - \ln(8)}$$

$\approx$ 9 miles

Example 5: A 800 gallon tank initially contains 700 gallons of brine containing 75 pounds of dissolved salt. Brine containing 4 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute and the well-stirred mixture flows out of the tank at the rate of 1 gallon per minute. Set up a differential equation for the amount of salt, $A(t)$, in the tank at time t.



Define: $V(t)$ = amount of brine mixture in tank at time t (in gallons)

$A(t)$ = amount of salt in the tank at time t

t = time in minutes

$$\frac{dA}{dt} = \begin{pmatrix}\text{rate of change of salt}\\ \text{in tank in lbs/min}\end{pmatrix} = \begin{pmatrix}\text{rate in}\\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{rate out}\\ \text{of salt}\end{pmatrix}$$

Rate in: $\left(4\ \frac{\text{lbs}}{\text{gallons}}\right)\left(4\ \frac{\text{gallons}}{\text{min}}\right) = 16\ \frac{\text{lbs}}{\text{min}}$

Rate out: "Well-stirred" means each gallon in the tank has as much salt in it as any other gallon.

i.e. Salt is uniformly mixed in the brine mixture

$$= \left(\frac{A(t)}{V(t)}\ \frac{\text{lbs}}{\text{gallons}}\right)\left(1\ \frac{\text{gallons}}{\text{min}}\right) = \frac{A(t)}{V(t)}\ \frac{\text{lbs}}{\text{min}}$$

$\frac{dA}{dt} = 16 - \frac{A(t)}{V(t)}$. Now find $V(t)$,

So $$\frac{dV}{dt} = \begin{pmatrix}\text{rate of change of}\\ \text{brine mix in gallons/min}\end{pmatrix} = \begin{pmatrix}\text{rate in}\\ \text{of brine}\end{pmatrix} - \begin{pmatrix}\text{rate out}\\ \text{of brine}\end{pmatrix}$$

$$= 4\ \frac{\text{gallons}}{\text{min}} - 1\ \frac{\text{gallons}}{\text{min}} = 3\ \frac{\text{gallons}}{\text{min}}$$

Hence $\begin{cases} V'(t) = 3 \\ V(0) = 700 \end{cases}$

So $V(t) = \int V'(t)\,dt = \int 3\,dt = 3t + C$

When $V(0) = 700$,

$700 = V(0) = 3(0) + C$

$700 = C \quad \Rightarrow V(t) = 3t + 700$

Hence $\frac{dA}{dt} = 16 - \frac{A}{3t + 700}$