Lesson 21: First-Order Linear Differential Equations I

Warm-Up: Compute the following derivatives:

ⓐ $\frac{d}{dt}\left(\int P(t)\,dt\right)$

By the Fundamental Theorem of Calculus (FTC),

$$\frac{d}{dt}\left(\int P(t)\,dt\right) = P(t)$$

ⓑ $\frac{d}{dt}\left(e^{\int P(t)\,dt}\right)$

By the Chain Rule,

$$\frac{d}{dt}\left(e^{\int P(t)\,dt}\right) = e^{\int P(t)\,dt} \cdot \frac{d}{dt}\left(\int P(t)\,dt\right) \underset{\text{by (a)}}{=} e^{\int P(t)\,dt} \cdot P(t)$$

ⓒ $\frac{d}{dt}\left(y(t)\,e^{\int P(t)\,dt}\right)$

By the Product Rule,

$$\frac{d}{dt}\left(y(t)\,e^{\int P(t)\,dt}\right) = y'(t)\,e^{\int P(t)\,dt} + y(t) \cdot \frac{d}{dt}\left(e^{\int P(t)\,dt}\right) \underset{\text{by (b)}}{=} y'(t)\,e^{\int P(t)\,dt} + y(t)P(t)\,e^{\int P(t)\,dt}$$

Let's define what a First-Order Linear Differential Equation

- First-order means that only the first derivative appears (so, no y'', y''', etc)
- Linear means that y' and y are not multiplied together in any combination.

<u>example</u>: $y' + ty = t^2 + 6$

<u>Not example</u>: $yy' + y = 1$

- Differential Equation is an equation that relates one or more functions and their derivatives.

A first order linear equation can be written in the standard form:

$$y' + P(t)y = Q(t) \quad (*)$$

Why do we want it in this form? To do the following

Let $u(t) = \exp[\int P(t)\,dt]$. Multiply both sides of $(*)$ by $u(t)$, we get

$$y'\exp[\int P(t)\,dt] + P(t)y\exp[\int P(t)\,dt] = Q(t)\exp[\int P(t)\,dt]$$

By Warm-Up ③, the LHS is

$$\frac{d}{dt}\left(y\exp[\int P(t)\,dt]\right) = Q(t)\exp[\int P(t)\,dt]$$

Remember $u(t) = \exp[\int P(t)\,dt]$

$$\frac{d}{dt}(y\cdot u(t)) = Q(t)u(t)$$

Integrate both sides by dt

$$\int \frac{d}{dt}(y\cdot u(t))\,dt = \int Q(t)u(t)\,dt$$

By the FTC,

$$y\cdot u(t) = \int Q(t)u(t)\,dt$$

Definition: The term $u(t) = \exp[\int P(t)\,dt]$ is called an integrating factor.

To summarize:

Given an equation of the form

$$y' + P(t)y = Q(t)$$

a solution is given by

$$y\cdot u(t) = \int Q(t)\,u(t)\,dt$$

where $u(t) = \exp[\int P(t)\,dt]$

How to solve First-Order Linear Equations

① Using simple algebra, rewrite your equation to be

$$y' + P(t) \cdot y = Q(t)$$

i.e. We are getting the equation into Standard Form.

② Determine $P(t)$ and $Q(t)$

③ Compute the integrating factor.

$$u(t) = \exp\left[\int P(t)\,dt\right]$$

④ Plug $u(t)$ and $Q(t)$ in

$$y \cdot u(t) = \int Q(t) \cdot u(t)\,dt + C$$

⑤ Integrate the RHS of 4.

⑥ Divide both sides of the equation from ⑤ by $u(t)$

In today's lecture, Step ① will be done.

Example 1: Find the general solution of

ⓐ $\frac{dy}{dx} + 11y = 5$

Since the ode in standard form, ① ✓

② $P(x) = 11 \qquad Q(x) = 5$

③ $u(x) = \exp\left[\int 11\,dx\right] = \exp[11x] = e^{11x}$

④ $y \cdot u(x) = \int Q(x)u(x)\,dx$

$y \cdot e^{11x} = \int 5e^{11x}\,dx$

⑤ $y e^{11x} = \int 5 e^{11x}\, dx$

$y e^{11x} = \frac{5}{11} e^{11x} + C$

⑥ $y = \frac{5}{11} \frac{e^{11x}}{e^{11x}} + \frac{C}{e^{11x}}$

$y = \frac{5}{11} + Ce^{-11x}$

ⓑ $\frac{dy}{dx} + \left(\frac{2}{x}\right) y = 3x - 5$

Since the ode in standard form, ① ✓

② $P(x) = \frac{2}{x}$ $\quad Q(x) = 3x - 5$

③ $u(x) = \exp\left[\int P(x) dx\right] = \exp\left[\int \frac{2}{x} dx\right]$

$= \exp[2 \ln x] = \exp[\ln x^2] = x^2$

④ $y \cdot u(x) = \int Q(x) u(x)\, dx$

$y \cdot x^2 = \int (3x - 5) x^2\, dx$

⑤ $y \cdot x^2 = \int (3x^3 - 5x^2)\, dx$

$y \cdot x^2 = \frac{3x^4}{4} - \frac{5x^3}{3} + C$

⑥ $y = \frac{3}{4} \frac{x^4}{x^2} - \frac{5}{3} \frac{x^3}{x^2} + \frac{C}{x^2}$

$= \frac{3}{4} x^2 - \frac{5}{3} x + \frac{C}{x^2}$

ⓒ $y' - y = 19$

Since the ode in standard form, ① ✓

② $P(x) = -1$ $\quad Q(x) = 19$

③ $u(x) = \exp[\int P(x)\,dx] = \exp[\int -1\,dx] = \exp[-x]$
$= e^{-x}$

④ $y \cdot u(x) = \int Q(x)\,u(x)\,dx$
$y \cdot e^{-x} = \int 19e^{-x}\,dx$

⑤ $y \cdot e^{-x} = \int 19e^{-x}\,dx$
$y \cdot e^{-x} = -19e^{-x} + C$

⑥ $y = -19\dfrac{e^{-x}}{e^{-x}} + \dfrac{C}{e^{-x}}$
$= -19 + Ce^{x}$

ⓓ $\dfrac{dy}{dx} + 9y = 3e^{-9x}$

Since the ode in standard form, ① ✓

② $P(x) = 9$ $Q(x) = 3e^{-9x}$

③ $u(x) = \exp[\int P(x)\,dx] = \exp[\int 9\,dx] = \exp[9x] = e^{9x}$

④ $y \cdot u(x) = \int Q(x)\,u(x)\,dx$
$y \cdot e^{9x} = \int 3e^{-9x}e^{9x}\,dx$
$y \cdot e^{9x} = \int 3\,dx$

⑤ $y \cdot e^{9x} = \int 3\,dx$
$y \cdot e^{9x} = 3x + C$

⑥ $y = (3x + C)e^{-9x}$