

Lesson 22: First-Order Linear Differential Equations II

Recall from Last Time, given an equation of the form

$$y' + P(t)y = Q(t)$$

a solution is given by

$$y \cdot u(t) = \int Q(t)u(t) dt$$

$$\text{where } u(t) = \exp\left[\int P(t) dt\right]$$

How to solve First-Order Linear Equations

① Using simply algebra, rewrite your equation to be

$$y' + P(t)y = Q(t)$$

i.e. We are getting the equation into Standard Form.

② Determine $P(t)$ and $Q(t)$

③ Compute integrating factor. i.e. $u(t) = \exp\left[\int P(t) dt\right]$

④ Plug $u(t)$ and $Q(t)$ in $y \cdot u(t) = \int Q(t)u(t) dt$

⑤ Integrate the RHS of ④.

⑥ Divide both sides of the equation from ⑤ by $u(t)$.

Example 1: Find the general solution of the following equations:

$$\textcircled{a} \quad t \frac{dy}{dt} + 12y = \frac{1}{t}$$

$$\textcircled{1} \quad t \frac{dy}{dt} + 12y = \frac{1}{t} \cdot \frac{1}{t}$$

$$\frac{dy}{dt} + \frac{12}{t}y = \frac{1}{t^2}$$

$$\textcircled{2} \quad P(t) = \frac{12}{t} \quad Q(t) = \frac{1}{t^2}$$

$$\begin{aligned} \textcircled{3} \quad u(t) &= \exp \left[\int P(t) dt \right] = \exp \left[\int \frac{12}{t} dt \right] \\ &= \exp \left[12 \ln(t) \right] = \exp \left[\ln(t^{12}) \right] = t^{12} \end{aligned}$$

$$\begin{aligned} \textcircled{4} \quad y \cdot u(t) &= \int Q(t) u(t) dt \\ y \cdot t^{12} &= \int \frac{1}{t^2} \cdot t^{12} dt \\ y \cdot t^{12} &= \int t^{10} dt \end{aligned}$$

$$\textcircled{5} \quad y \cdot t^{12} = \frac{t^{11}}{11} + C$$

$$\begin{aligned} \textcircled{6} \quad y &= \frac{1}{t^{12}} \cdot \frac{t^{11}}{11} + \frac{C}{t^{12}} \\ &= \frac{1}{11t} + \frac{C}{t^{12}} \end{aligned}$$

$$\textcircled{b} \quad 2y' + 10y = 20e^{-10x}$$

$$\begin{aligned} \textcircled{1} \quad \frac{2y'}{2} + \frac{10y}{2} &= \frac{20e^{-10x}}{2} \\ y' + 5y &= 10e^{-10x} \end{aligned}$$

$$\textcircled{2} \quad P(x) = 5 \quad Q(x) = 10e^{-10x}$$

$$\textcircled{3} \quad u(x) = \exp \left[\int P(x) dx \right] = \exp \left[\int 5 dx \right] = e^{5x}$$

$$\begin{aligned} \textcircled{4} \quad y \cdot u(x) &= \int Q(x) u(x) dx \\ y \cdot e^{5x} &= \int 10e^{-10x} e^{5x} dx \\ y \cdot e^{5x} &= \int 10e^{-5x} dx \end{aligned}$$

$$\textcircled{5} y \cdot e^{5x} = \frac{10}{-5} e^{-5x} + C$$

$$y \cdot e^{5x} = -2e^{-5x} + C$$

$$\textcircled{6} y = \frac{-2e^{-5x}}{e^{+5x}} + \frac{C}{e^{5x}}$$

$$y = -2e^{-10x} + Ce^{-5x}$$

$$\textcircled{c} (x-4)y' + y = x^2 + 1$$

$$\textcircled{1} \frac{(x-4)y'}{x-4} + \frac{1}{x-4}y = \frac{x^2+1}{x-4}$$

$$y' + \frac{1}{x-4}y = \frac{x^2+1}{x-4}$$

$$\textcircled{2} P(x) = \frac{1}{x-4} \quad Q(x) = \frac{x^2+1}{x-4}$$

$$\textcircled{3} u(x) = \exp\left[\int P(x) dx\right] = \exp\left[\int \frac{1}{x-4} dx\right]$$
$$= \exp[\ln(x-4)] = x-4$$

$$\textcircled{4} y \cdot u(x) = \int Q(x) u(x) dx$$

$$y \cdot (x-4) = \int \frac{x^2+1}{x-4} \cdot (x-4) dx$$

$$y \cdot (x-4) = \int (x^2+1) dx$$

$$\textcircled{5} y \cdot (x-4) = \frac{x^3}{3} + x + C$$

$$\textcircled{6} y = \frac{x^3}{3(x-4)} + \frac{x}{x-4} + \frac{C}{x-4}$$

$$(d) 3x^2 y + x^3 y' = 7 \sec^2 x \tan x$$

$$(1) x^3 y' + 3x^2 y = 7 \sec^2 x \tan x$$

$$\frac{x^3 y'}{x^3} + \frac{3x^2 y}{x^3} = \frac{7 \sec^2 x \tan x}{x^3}$$

$$y' + \frac{3}{x} y = \frac{7 \sec^2 x \tan x}{x^3}$$

$$(2) P(x) = \frac{3}{x} \quad Q(x) = \frac{7 \sec^2 x \tan x}{x^3}$$

$$(3) u(x) = \exp\left[\int P(x) dx\right] = \exp\left[\int \frac{3}{x} dx\right] = \exp[3 \ln(x)] \\ = \exp[\ln(x^3)] = x^3$$

$$(4) y \cdot u(x) = \int Q(x) u(x) dx \\ y \cdot x^3 = \int \frac{7 \sec^2 x \tan x}{x^3} \cdot x^3 dx$$

$$y \cdot x^3 = \int 7 \sec^2 x \tan x dx$$

$$(5) \quad u = \tan x \quad du = \sec^2 x dx \\ y \cdot x^3 = \int 7 u du$$

$$y \cdot x^3 = \frac{7u^2}{2} + C$$

$$y \cdot x^3 = \frac{7}{2} \tan^2 x + C$$

$$(6) y = \frac{7}{2} \frac{\tan^2 x}{x^3} + \frac{C}{x^3}$$

$$(e) x^4 y' + x^3 y = 4x^4 \cos(x)$$

$$(1) \frac{x^4 y'}{x^4} + \frac{x^3 y}{x^4} = \frac{4x^4 \cos(x)}{x^4}$$

$$y' + \frac{1}{x} y = 4 \cos(x)$$

$$\textcircled{2} P(x) = \frac{1}{x} \quad Q(x) = 4\cos(x)$$

$$\textcircled{3} u(x) = \exp\left[\int P(x)dx\right] = \exp\left[\int \frac{1}{x} dx\right] = \exp[\ln x] = x$$

$$\textcircled{4} y \cdot u(x) = \int Q(x)u(x)dx$$
$$y \cdot x = \int 4\cos(x) \cdot x dx$$

$$\textcircled{5} \quad \begin{array}{ll} u = x & dv = 4\cos(x) dx \\ du = dx & v = 4\sin x \end{array}$$

$$y \cdot x = uv - \int v du$$

$$y \cdot x = 4x\sin x - \int 4\sin x dx$$

$$y \cdot x = 4x\sin x + 4\cos x + C$$

$$y = \frac{4x\sin x}{x} + \frac{4\cos x}{x} + \frac{C}{x}$$

$$y = 4\sin x + \frac{4\cos x}{x} + \frac{C}{x}$$