Lesson 25: Maclaurin Series

· A special case of a Power Series is called a Maclaurin Series.

Definition: A Maclaurin Series has the form

$$f(x)=\sum_{n=0}^{\infty} c_n x^n \text{ where } c_n=\frac{f^{(n)}(0)}{n!}, \text{ and } |x|<R$$

In other words,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\dots$$

Let's see how to use this.

Example 1: Find the Maclaurin series representation of $f(x)=\sin x$

Start by getting the derivatives and evaluate at $x=0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$f(x)=\sin x$	$f(0)=0$
1	$f'(x)=\cos x$	$f'(0)=1$
2	$f''(x)=-\sin x$	$f''(0)=0$
3	$f'''(x)=-\cos x$	$f'''(0)=-1$
4	$f^{(4)}(x)=\sin x$	$f^{(4)}(0)=0$
5	$f^{(5)}(x)=\cos x$	$f^{(5)}(0)=1$
6	$f^{(6)}(x)=-\sin x$	$f^{(6)}(0)=0$
7	$f^{(7)}(x)=-\cos x$	$f^{(7)}(0)=-1$

So $f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}+\dots$

$$=0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{-1}{3!}x^3+\frac{0}{4!}x^4+\frac{1}{5!}x^5+\frac{0}{6!}x^6+\frac{-1}{7!}x^7+\dots$$

$$=\frac{1}{1!}x-\frac{1}{3}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\dots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=\sin(x)$$

The Maclaurin series for other function can be found in a similar fashion. We will be kind enough to given them to you now and on the exam.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example 2: Find a power series expansion for

ⓐ $f(x) = \ln(1+4x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Replace x by 4x in the formula.

$$\ln(1+4x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(4x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4^n}{n} x^n$$

ⓑ $f(x) = e^{-3x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Replace x by $-3x^2$ in the formula

$$e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!} x^{2n}$$

ⓒ $f(x) = 4x\cos(2x)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Replace x by 2x in the formula

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n}$$

Multiply by 4x

$$4x\cos(2x) = 4x \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{n+1}}{(2n)!} x^{2n+1}$$

Estimating Function Values Using Power Series

- Power series can be used to estimate a function's value at a given value of x.

Example 4: Estimate the value of ln(1.62) using the first four terms of the Maclaurin series for $f(x) = \ln(1+x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \underbrace{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}_{\text{First four terms}} + \ldots$$

Note that this is for $\ln(x+1)$ and we want $\ln(1.62)$. So

$$x+1 = 1.62 \rightarrow x = 0.62$$

So at $x = 0.62$,

$$(0.62) - \frac{(0.62)^2}{2} + \frac{(0.62)^3}{3} - \frac{(0.62)^4}{4} \approx 0.47030$$

Example 3: Use the first three nonzero terms of the Maclaurin series for $f(x) = \sin x$ to estimate $\sin(0.49)$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \underbrace{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}}_{\text{First three terms}} - \ldots$$

So at $x = 0.49$,

$$\frac{(0.49)}{1!} - \frac{(0.49)^3}{3!} + \frac{(0.49)^5}{5!} \approx 0.470627$$

Using Maclaurin Series to Estimate Definite Integrals

Steps to Solve Integrals Using Maclaurin

1. Convert the function into a series.
2. Integrate the series (remember x is the variable)
3. Write out the number of terms to be used.
4. Substitute in the bounds.

Example 5: Use the first 4 terms of the Maclaurin series to estimate the integral

$$\int_0^{0.55} \cos(\sqrt{x})\,dx$$

① Write $\cos(x^{1/2})$ as a Maclaurin Series

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Replace x with $x^{1/2}$.

$$\cos(x^{1/2}) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{1/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$$

② Integrate

$$\int \cos(x^{1/2})\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{x^{n+1}}{n+1}$$

③ Write out 4 first terms.

$n=0$: $\frac{(-1)^0}{(2\cdot 0)!} \cdot \frac{x^{0+1}}{0+1} = x$

$n=1$: $\frac{(-1)^1}{(2\cdot 1)!} \cdot \frac{x^{1+1}}{1+1} = -\frac{x^2}{4}$

$n=2$: $\frac{(-1)^2}{(2\cdot 2)!} \cdot \frac{x^{2+1}}{2+1} = \frac{x^3}{72}$

$n=3$: $\frac{(-1)^3}{(2\cdot 3)!} \cdot \frac{x^{3+1}}{3+1} = -\frac{x^4}{2880}$

④ Evaluate ③ from 0 to 0.55

$$\left(0.55 - \frac{(0.55)^2}{4} + \frac{(0.55)^3}{72} - \frac{(0.55)^4}{2880}\right) - 0 \approx 0.47665$$