Lesson 28: Higher Order Partial Derivatives

Recall from last class, the partial derivatives of $z = f(x,y)$ are

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

Higher Order Partial Derivatives are similar to Higher Order Derivatives (Lesson 13) from Applied Calculus I. The main difference is that we can mix partials (i.e. first do x then y or vice versa).

Notation/Definition: Taking the partial derivatives of the partials f_x and f_y, we have the following second-order partials.

1. $\frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
2. $\frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$
3. $\frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$
4. $\frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Fact: While not true in general, but for the purpose of this class, we have

$$f_{xy} = f_{yx}$$

So ② and ③, in the definition above, can be combined

Example 1: Compute the second order partial derivatives

ⓐ $f(x,y) = 5x^2y + 2xy^3 + 3y^2$

First find f_x and f_y.

$$f_x = 10xy + 2y^3 \qquad f_y = 5x^2 + 6xy^2 + 6y$$

Next find f_{xx}, f_{xy}, and f_{yy} via the definition.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(10xy + 2y^3) = 10y + 0 = 10y$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(10xy + 2y^3) = 10x + 6y^2$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(5x^2 + 6xy^2 + 6y) = 0 + 12xy + 6 = 12xy + 6$$

ⓑ $f(x,y) = x^2 y e^{7x}$

First find f_x and f_y.

$$f(x,y) = y(x^2 e^{7x})$$

$$f_x = y(2xe^{7x} + x^2 e^{7x} \cdot 7) = ye^{7x}(2x + 7x^2)$$

Product Rule

$$f(x,y) = (x^2 e^{7x})y$$

$$f_y = x^2 e^{7x}$$

Next find f_{xx}, f_{xy}, and f_{yy} via the definition.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(ye^{7x}(2x+7x^2)\right) = y\frac{\partial}{\partial x}\left(e^{7x}(2x+7x^2)\right)$$

$$= y\left(7e^{7x}(2x+7x^2) + e^{7x}(2+14x)\right)$$

$$= ye^{7x}[7(2x+7x^2) + (2+14x)] = ye^{7x}[14x + 49x^2 + 2 + 14x]$$

$$= ye^{7x}(49x^2 + 28x + 2)$$

$$f_{xy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(ye^{7x}(2x+7x^2)) = e^{7x}(2x+7x^2)\frac{d}{dy}(y)$$

$$= e^{7x}(2x+7x^2)$$

$$f_{yy} = \frac{\partial}{\partial y}(x^2 e^{7x}) = 0$$ b/c there is no y terms

ⓒ $f(x,y) = \ln(6x^2 + 7y)$

Chain Rule $\rightarrow$ First find f_x and f_y.

$$f_x = \frac{1}{6x^2+7y} \cdot \frac{\partial}{\partial x}(6x^2+7y) = \frac{12x}{6x^2+7y}$$

$$f_y = \frac{1}{6x^2+7y} \cdot \frac{\partial}{\partial y}(6x^2+7y) = \frac{7}{6x^2+7y}$$

Next find f_{xx}, f_{xy}, and f_{yy} via the definition.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{12x}{6x^2+7y}\right) = \frac{12(6x^2+7y) - 12x(12x+0)}{(6x^2+7y)^2}$$

$$= \frac{72x^2 + 84y - 144x^2}{(6x^2+7y)^2}$$

$$f_x = 12x(6x^2+7y)^{-1}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = 12x \cdot (-1)(6x^2+7y)^{-2} \cdot (0+7) = \frac{-84x}{(6x^2+7y)^2}$$

$$f_y = 7(6x^2+7y)^{-1}$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = 7 \cdot (-1)(6x^2+7y)^{-2} \cdot (0+7) = \frac{-49}{(6x^2+7y)^2}$$