

Lesson 31: La Grange Multipliers

Constrained Min/Max

Previously, we optimized a function on a region that contained its boundary via the **Second Derivative Test for Multi-variable Functions**. However, as we saw in the examples finding potential optimal points on the boundary was LONG and MESSY process.

Now we are going to take a look at another way of optimizing a function, $f(x,y)$, subject to the given constraint $g(x,y)=k$.

Method of LaGrange Multipliers

① Solve the following system of equations.

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$$

② Evaluate f at every point (x,y) found in ①.

↳ The largest is the maximum value of f ,

↳ The smallest is the minimum value of f ,

Note the equations from ① are known as LaGrange Equations and the constant, λ , is called the LaGrange Multiplier.

Example 1: Find the minimum value of $f(x,y) = x^2 + y^2$ subject to the constraint $7y = 6 - 2x$

Here $g(x,y) = 7y + 2x = 6$. Using LaGrange Multipliers,

$$\begin{array}{ll} f_x = 2x & \lambda g_x = \lambda 2 \\ f_y = 2y & \lambda g_y = \lambda 7 \end{array}$$

So our system is $\begin{cases} 2x = \lambda 2 \\ 2y = \lambda 7 \\ 7y + 2x = 6 \end{cases} \Rightarrow \begin{cases} x = \lambda & \text{①} \\ y = (\frac{7}{2})\lambda & \text{②} \\ 7y + 2x = 6 & \text{③} \end{cases}$

Plug ① and ② into ③.

$$7\left(\frac{7}{2}\lambda\right) + 2\lambda = 6$$

$$\frac{49}{2}\lambda + 2\lambda = 6$$

$$49\lambda + 4\lambda = 12$$

$$53\lambda = 12$$

$$\lambda = \frac{12}{53}$$

$$\frac{12}{53}$$

Plug $\lambda = 12/53$ into ① and ②.

$$\textcircled{1} x = \lambda = \frac{12}{53}$$

$$\textcircled{2} y = \frac{7}{2}\lambda = \frac{7}{2} \cdot \frac{12}{53} = \frac{42}{53}$$

$$\text{So } f\left(\frac{12}{53}, \frac{42}{53}\right) = \left(\frac{12}{53}\right)^2 + \left(\frac{42}{53}\right)^2 \approx 0.6792$$

Since there is only one value, 0.6792 is the min.

Example 2: Find the maximum value of $f(x,y) = 31x^{3/2}y$ subject to the constraint $x+y=46$

Here $g(x,y) = x+y=46$. Using La Grange Multipliers,

$$\textcircled{1} f_x = 31\left(\frac{3}{2}\right)x^{1/2}y = \frac{93}{2}x^{1/2}y \quad \lambda g_x = \lambda(1)$$

$$f_y = 31x^{3/2}$$

$$\lambda g_y = \lambda(1)$$

So our system is

$$\begin{cases} \frac{93}{2}x^{1/2}y = \lambda & \textcircled{1} \\ 31x^{3/2} = \lambda & \textcircled{2} \\ x+y = 46 & \textcircled{3} \end{cases}$$

Set ① and ② equal.

$$\frac{93}{2}x^{1/2}y = \lambda = 31x^{3/2}$$

$$93x^{1/2}y = 62x^{3/2}$$

$$93x^{1/2}y - 62x^{3/2} = 0$$

$$31x^{1/2}(3y - 2x) = 0$$

$$x=0$$

$$y = \frac{2x}{3}$$

Plug $x=0$ into ③,

$$0+y=46$$

$$y=46$$

$$(0, 46)$$

Plug $y = \frac{2}{3}x$ into (3)

$$x + \left(\frac{2}{3}x\right) = 46$$

$$\frac{5x}{3} = 46$$

$$x = \frac{138}{5}$$

$$\left(\frac{138}{5}, \frac{92}{5}\right)$$

Testing for Max

$$f(x, y) = 31x^{3/2}y$$

$$f(0, 46) = 0$$

$$f\left(\frac{138}{5}, \frac{92}{5}\right) \approx 82707 \Rightarrow \text{max}$$

Example 3: Find the maximum value of the function $f(x, y) = 3x - 11y^2$ subject to the constraint $x^2 + 11y^2 = 81$.

Here $g(x, y) = x^2 + 11y^2 = 81$. Using Lagrange Multipliers,

$$f_x = 3$$

$$f_y = -22y$$

$$\lambda g_x = \lambda(2x)$$

$$\lambda g_y = \lambda(22y)$$

So our system is

$$\begin{cases} 3 = 2\lambda x \\ -22y = \lambda 22y \\ x^2 + 11y^2 = 81 \end{cases} \Rightarrow \begin{cases} 3 = 2\lambda x & (1) \\ \lambda 22y + 22y = 0 & (2) \\ x^2 + 11y^2 = 81 & (3) \end{cases}$$

Solve (2).

$$\lambda 22y + 22y = 0$$

$$22y(\lambda + 1) = 0$$

$$y = 0, \lambda = -1$$

Plug $y = 0$ into (3)

$$x^2 + 11(0)^2 = 81$$

$$x^2 = 81$$

$$x = \pm 9$$

$$(9, 0), (-9, 0)$$

Plug $\lambda = -1$ into (2)

$$3 = 2(-1)x$$

$$3 = -2x$$

$$x = -3/2$$

Plug $x = -3/2$ into (3)

$$\left(-\frac{3}{2}\right)^2 + 11y^2 = 81$$

$$9/4 + 11y^2 = 81$$

$$11y^2 = 315/4$$

$$y^2 = 315/44$$

$$y = \pm \sqrt{\frac{315}{44}}$$

So points to test are:

$$\bullet (9, 0)$$

$$\bullet (-9, 0)$$

$$\bullet \left(-\frac{3}{2}, \sqrt{\frac{315}{44}}\right)$$

$$\bullet \left(-\frac{3}{2}, -\sqrt{\frac{315}{44}}\right)$$

Testing for Max

$$f(x,y) = 3x - 11y^2$$

$$f(9,0) = 27 \Rightarrow \text{max}$$

$$f(-9,0) = -27$$

$$f\left(\frac{-3}{2}, \sqrt{\frac{315}{44}}\right) = -\frac{333}{4}$$

$$f\left(\frac{-3}{2}, -\sqrt{\frac{315}{44}}\right) = -\frac{333}{4}$$

Example 4: Find the minimum and maximum of the function $f(x,y) = e^{-xy}$ subject to the constraint $5x^2 + 19y^2 = 17$.

Here $g(x,y) = 5x^2 + 19y^2 = 17$. Using Lagrange Multipliers,

$$f_x = -ye^{-xy}$$

$$\lambda g_x = \lambda(10x)$$

$$f_y = -xe^{-xy}$$

$$\lambda g_y = \lambda(38y)$$

So our system is

$$\begin{cases} -ye^{-xy} = 10\lambda x & \textcircled{1} \\ -xe^{-xy} = 38\lambda y & \textcircled{2} \\ 5x^2 + 19y^2 = 17 & \textcircled{3} \end{cases}$$

Multiply e^{-xy} to both $\textcircled{1}$ and $\textcircled{2}$.

$$\begin{cases} -y = 10\lambda x e^{-xy} & \textcircled{1} \\ -x = 38\lambda y e^{-xy} & \textcircled{2} \\ 5x^2 + 19y^2 = 17 & \textcircled{3} \end{cases}$$

Next, multiply $\textcircled{1}$ by $38y$ and $\textcircled{2}$ by $10x$.

$$\begin{cases} -38y^2 = 380\lambda x y e^{-xy} & \textcircled{1} \\ -10x^2 = 380\lambda x y e^{-xy} & \textcircled{2} \\ 5x^2 + 19y^2 = 17 & \textcircled{3} \end{cases}$$

Note the RHS of $\textcircled{1}$ and $\textcircled{2}$ match. So set $\textcircled{1} = \textcircled{2}$.

$$-38y^2 = -10x^2$$

$$19y^2 = 5x^2$$

$$y = \pm \sqrt{\frac{5}{19}} x$$

Plug $5x^2 = 14y^2$ into ③

$$5x^2 + 5x^2 = 17$$

$$10x^2 = 17$$

$$x^2 = \pm \sqrt{\frac{17}{10}}$$

$$\text{When } x = \sqrt{\frac{17}{10}}, y = \pm \sqrt{\frac{5}{19}} \cdot \sqrt{\frac{17}{10}} = \pm \sqrt{\frac{17}{38}}$$

$$\text{which gives us } \left(\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}} \right), \left(\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}} \right)$$

$$\text{When } x = -\sqrt{\frac{17}{10}}, y = \pm \sqrt{\frac{5}{19}} \left(-\sqrt{\frac{17}{10}} \right) = \mp \sqrt{\frac{17}{38}}$$

$$\text{which gives us } \left(-\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}} \right), \left(-\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}} \right)$$

Test for min/max

$$f\left(\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}}\right) = 0.41 \Rightarrow \text{min}$$

$$f\left(\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}}\right) = 2.39 \Rightarrow \text{max}$$

$$f\left(-\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}}\right) = 2.39 \Rightarrow \text{max}$$

$$f\left(-\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}}\right) = 0.41 \Rightarrow \text{min}$$