Lesson 35: Double Integrals III

Recall from Lesson 2, the formula for average value:

For $f(x)$ defined on $[a,b]$, the average value of $f(x)$ on $[a,b]$ is

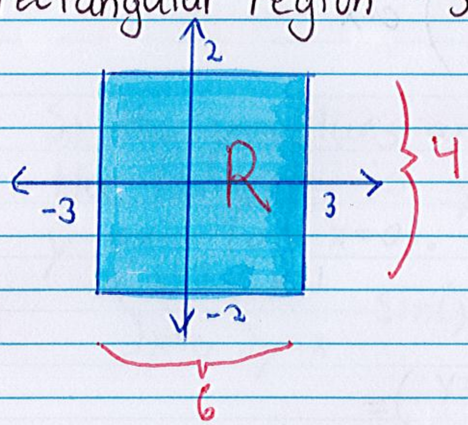
$$f_{AVE}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The multivariable average value formula follows:

For $f(x,y)$ defined on a region, R, the average value of $f(x,y)$ over the region, R, is given by

$$f_{AVE}(x,y) = \frac{1}{A}\iint_R f(x,y)\,dA$$ where A is the area of R.

Example 1: Find the average of $f(x,y) = 12 - x^2 - y^2$ in a rectangular region $-3 \le x \le 3$, $-2 \le y \le 2$.



First draw the region. With the drawing, find the area of R.

Area of R $= 6 \times 4 = 24$

Note we are given the bounds for our integral. So

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\int_{y=-2}^{y=2}(12-x^2-y^2)\,dy\,dx$$

Now integrate.

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\left(\left(12y - x^2y - \frac{y^3}{3}\right)\right]_{y=-2}^{y=2} dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(12(2) - 2x^2 - \frac{2^3}{3} - \left(12(-2) - x^2(-2) - \frac{(-2)^3}{3}\right)\right) dx$$

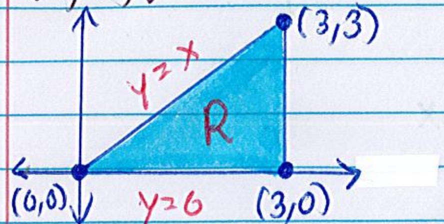
$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(24 - 2x^2 - \frac{8}{3} + 24 - 2x^2 - \frac{8}{3}\right) dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(\frac{128}{3} - 4x^2\right) dx$$

$$= \frac{1}{24}\left(\frac{128}{3}x - \frac{4x^3}{3}\right)\Bigg]_{x=-3}^{x=3}$$

$$= \frac{1}{24}\left(\frac{128}{3}(3) - \frac{4(3)^3}{3} - \left(\frac{128}{3}(-3) - \frac{4(-3)^3}{3}\right)\right) = \frac{28}{3}$$

Example 2: Find the average value of $f(x,y) = x^2 + 2xy + y^2$ in the triangular region with vertices $(0,0)$, $(3,0)$, and $(3,3)$.



First draw the region, with drawing, find the area of R,

$$\text{Area of } R = \frac{1}{2}(3)(3) = \frac{9}{2}$$

Next, we need to determine the bounds for our integral. The x-values are $0 \le x \le 3$. As for y, we see that y lies between the x-axis ($y=0$) and the line $y=x$. So

$$f_{AVE} = \frac{1}{9/2}\int_{x=0}^{x=3}\int_{y=0}^{y=x}(x^2 + 2xy + y^2)\,dy\,dx$$

$$= \frac{2}{9}\int_{x=0}^{x=3}\left(\left(x^2 \cdot y + \frac{2xy^2}{2} + \frac{y^3}{3}\right)\Bigg]_{y=0}^{y=x}\right)dx$$

$$= \frac{2}{9}\int_{x=0}^{x=3}\left(x^3 + x \cdot x^2 + \frac{x^3}{3}\right)dx$$

$$= \frac{2}{9}\int_{x=0}^{x=3}\frac{7}{3}x^3\,dx$$

$$= \frac{2}{9}\cdot\frac{7}{3}\,\frac{x^4}{4}\Bigg]_{x=0}^{x=3}$$

$$= \frac{2}{9}\cdot\frac{7}{3}\cdot\frac{3^4}{4} = \frac{21}{2}$$